

# MATRIX ESTIMATION BY UNIVERSAL SINGULAR VALUE THRESHOLDING

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**ABSTRACT.** Consider the problem of estimating the entries of a large matrix, when the observed entries are noisy versions of a small random fraction of the original entries. This problem has received widespread attention in recent times, especially after the pioneering works of Emmanuel Candès and collaborators. Typically, it is assumed that the underlying matrix has low rank. This paper introduces a simple estimation procedure, called Universal Singular Value Thresholding (USVT), that works for any matrix that has ‘a little bit of structure’. In particular, the matrix need not be of low rank. The procedure is very simple and fast, works under minimal assumptions, and is applicable for very large matrices. Surprisingly, this simple estimator achieves the minimax error rate up to a constant factor. The method is applied to give simple solutions to difficult questions in low rank matrix estimation, blockmodels, distance matrix completion, latent space models, positive definite matrix completion, problems related to graph limits, and generalized Bradley-Terry models for pairwise comparison.

## CONTENTS

1. Introduction	2
2. The USVT algorithm and estimator	6
2.1. The setup	6
2.2. The USVT estimator	6
2.3. The main theorem	7
2.4. The minimax lower bound	8
2.5. A numerical result	8
2.6. Dependent entries	10
2.7. Plan of the paper	10
3. Comparison with the existing literature	10
4. Examples	12
4.1. Low rank matrices	12
4.2. The stochastic blockmodel	14

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4.3.	Distance matrices	15
4.4.	Latent space models	17
4.5.	Positive definite matrices	19
4.6.	Graphons	20
4.7.	Non-parametric Bradley-Terry model	21
4.8.	Structured matrices	24
5.	The model-free case: A connection with Szemerédi's lemma	26
6.	An impossibility theorem for error estimates	29
7.	Simulation results	32
7.1.	Blockmodels	32
7.2.	Latent space models	33
7.3.	Covariance matrix completion	33
7.4.	Distance matrix completion	34
7.5.	Collaborative filtering	35
7.6.	Further work	35
8.	Proof of Theorem 2.1	35
8.1.	Matrix norms	35
8.2.	Perturbation of singular values	36
8.3.	Bernstein's inequality	37
8.4.	Talagrand's concentration inequality	37
8.5.	Spectral norms of random matrices	38
8.6.	The estimation lemma	39
8.7.	Finishing the proof of Theorem 2.1	41
9.	Proof of Theorem 2.2	43
10.	Proof of Theorem 2.3	46
11.	Discussion and summary	46
	References	47

## 1. INTRODUCTION

Consider a statistical estimation problem where the unknown parameter is not a single value or vector, but an  $m \times n$  matrix  $M$ . Given an estimator  $\hat{M}$ , one choice for a measure of the error in estimation is the mean-squared error, defined as

$$(1) \quad \text{MSE}(\hat{M}) := \mathbb{E} \left[ \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (\hat{m}_{ij} - m_{ij})^2 \right].$$

Here  $\hat{m}_{ij}$  and  $m_{ij}$  denote the  $(i, j)$ th elements of  $\hat{M}$  and  $M$ , respectively. If we have a sequence of such problems, and  $M_n$  and  $\hat{M}_n$  denote the parameter and the estimator in the  $n$ th problem, then by usual statistical terminology we may say that the sequence of estimators  $\hat{M}_n$  is consistent if

$$\lim_{n \rightarrow \infty} \text{MSE}(\hat{M}_n) = 0.$$

To fix ideas, let us begin by posing a list of challenging matrix estimation problems. The background and literature for these problems will be discussed in detail in Section 4.

**Problem 1.** (Low rank matrices.) Suppose that  $x_{ij}$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, n$  is a bunch of independent random variables, modeled as

$$x_{ij} = m_{ij} + \epsilon_{ij},$$

where  $\epsilon_{ij}$ 's have mean zero and  $m_{ij}$ 's are unknown constants. Additionally, suppose that each  $x_{ij}$  is observed with probability  $p$  and unobserved with probability  $1 - p$ , where  $p$  is some number in  $[0, 1]$ . Note that  $p$  is allowed to be quite small, depending on  $m$  and  $n$ . Assume that the  $x_{ij}$ 's are uniformly bounded above and below by fixed constants. In general, it is impossible to estimate the  $m_{ij}$ 's using the  $x_{ij}$ 's, since there is exactly one observation for each parameter. However, if the rank  $r$  of the matrix  $M = (m_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$  is sufficiently small, one may hope to accurately estimate  $M$  from the observed  $x_{ij}$ 's. The question is: *What is a necessary and sufficient condition, in terms of  $r$ ,  $m$ ,  $n$  and  $p$ , under which this estimation problem is solvable?* Low rank matrix estimation has been tackled in a vast number of papers in recent times [10, 1, 46, 88, 92, 26, 25, 28, 24, 79, 65, 66, 68, 67, 81, 38]. We will consider this problem in Section 4.1 as the simplest application of our technique.

**Problem 2.** (Stochastic blockmodels.) Consider an undirected graph on  $n$  vertices. A stochastic blockmodel assumes that the vertices  $1, \dots, n$  are partitioned into  $k$  blocks, and the probability that a vertex  $i$  is connected to vertex  $j$  by an edge depends on the blocks to which  $i$  and  $j$  belong. The edges are independent of each other. Let  $M$  be the matrix whose  $(i, j)$ th element is the probability of an edge existing between vertices  $i$  and  $j$ . If the number of blocks is comparable to  $n$ , then one cannot hope to estimate the edge probabilities. The question is: *Given that we do not know which vertex belongs to which block, or the number of blocks, is it possible to accurately estimate  $M$  (in mean-squared error) from a single realization of the graph, simply under the condition that  $k$  is small compared to  $n$ ?* The stochastic blockmodel has received a lot of attention over several decades [58, 96, 82, 37, 16, 71, 90, 17, 36, 78, 91], but the above question has not been completely resolved. We will solve it in Section 4.2.

**Problem 3.** (Distance matrix completion.) Suppose that we have a compact metric space  $K$  with metric  $d$ . Let  $x_1, \dots, x_n$  be  $n$  points from  $K$ . Let  $M_n$  be the matrix whose  $(i, j)$ th entry is  $d(x_i, x_j)$ . Now suppose that we do not observe  $M_n$ , but only a small fraction of its entries. More precisely, suppose that there is a number  $p \in [0, 1]$  such that each entry of  $M_n$  is observed with probability  $p$  and unobserved with probability  $1 - p$ , independent of other entries. The set of observed entries of  $M_n$  is our data. The question

is: *Given that we do not know anything about the metric space  $K$  or the metric  $d$  (except that it is compact), or the points  $x_1, \dots, x_n$ , is it possible to consistently estimate  $M_n$  as  $n \rightarrow \infty$  in the sense of mean squared error?*

Distance matrix completion has been a popular topic in the social sciences, multidimensional scaling and various other fields [97, 11, 7, 18, 94, 95, 19] but a proper statistical theory has never been developed. In the Euclidean case it is solvable by low rank methods, but general distance matrix completion is an unsolved problem. We will give the solution of this problem in Section 4.3.

**Problem 4.** (Latent space models.) Suppose that our data is an undirected graph on  $n$  vertices. The following is an example of a latent space model for this data: To each vertex is attached a latent (or hidden) variable  $\beta_i$ , which is a vector in a compact set  $K \subseteq \mathbb{R}^k$  where  $k$  is the dimension of the parameter space. Assume that the edges are independent, and the probability that there exists an edge between two vertices  $i$  and  $j$  is  $f(\beta_i, \beta_j)$ , where  $f : K \times K \rightarrow \mathbb{R}$  is a continuous function. The parameter in this model is the  $n \times n$  matrix  $M_n$  whose  $(i, j)$ th entry is  $f(\beta_i, \beta_j)$ . The question is: *Given that we do not know the function  $f$ , the parameters  $\beta_1, \dots, \beta_n$ , the region  $K$ , or even the dimension  $k$ , is it still possible to consistently estimate the matrix  $M_n$  in the sense of mean-squared error as  $n \rightarrow \infty$ ?* Latent space models are used extensively in the world of network analysis, but rigorous results are scarce [57, 16, 17, 55, 5, 31]. In particular, not much is known about how to estimate the parameters if the function  $f$  is unknown. We will solve this general problem in Section 4.4.

**Problem 5.** (Positive definite matrix completion.) Suppose that we have a positive semidefinite matrix  $M_n$  of order  $n$  with all entries bounded by some fixed constant, and only a random fraction  $p$  of its entries are observed. *Is it possible to estimate the matrix  $M_n$  without assuming any additional condition such as low rank? What is the fastest that  $p$  can tend to zero as  $n \rightarrow \infty$  so that it is possible to estimate  $M_n$  consistently?* Indeed, it is not obvious that  $M_n$  may be consistently estimated even if  $p$  remains fixed. Positive definite matrix completion has been a popular topic in linear algebra [53, 63, 15] and more recently, in statistical matrix completion [25, 26, 28] in the context of covariance matrices. However, the completion of positive definite matrices that are not necessarily of low rank is an open problem. We will solve it in Section 4.5.

**Problem 6.** (Estimating graphons.) Suppose that  $f : [0, 1]^2 \rightarrow [0, 1]$  is a measurable function and  $U_1, \dots, U_n$  are i.i.d. Uniform $[0, 1]$  random variables. Suppose that a random graph  $G_n$  on  $n$  vertices is constructed by putting an edge between vertices  $i$  and  $j$  with probability  $f(U_i, U_j)$ . In modern graph theory, a function such as  $f$  is called a ‘graphon’ and  $G_n$  is called a

graph ‘sampled from the graphon  $f$ ’. Let  $M_n$  be the random matrix whose  $(i, j)$ th element is  $f(U_i, U_j)$ . The question is: *Given that we do not know the function  $f$  nor anything about the values of  $U_1, \dots, U_n$ , is it possible to consistently estimate  $M_n$  by only observing the graph  $G_n$ ?* This is similar to Problem 4, with the difference that the function  $f$  is assumed to be only measurable; continuity is not assumed. The study of graphons has been popular in the recent graph theory literature [20, 21, 22, 72, 73, 74] and has connections to the study of weakly exchangeable arrays [42, 9, 6, 59]. They have also appeared recently in large deviations [32, 33, 75] and mathematical statistics [30, 86]. The question of estimating a graphon using a single graph sampled from the graphon has not been tackled in the literature, perhaps because of the impossible-looking nature of the problem. We will solve this question in Section 4.6.

**Problem 7.** (Non-parametric Bradley-Terry model.) Suppose there are  $n$  teams playing against each other in a tournament. Every team plays against every other team exactly once. Suppose that  $p_{ij}$  is the probability that team  $i$  wins against team  $j$  in a game between  $i$  and  $j$ . The Bradley-Terry model assumes that  $p_{ij}$  is of the form  $a_i/(a_i + a_j)$  for some unknown nonnegative numbers  $a_1, \dots, a_n$ . It is known how to estimate the parameters  $a_1, \dots, a_n$  if we assume that the outcomes of all games are independent. Now suppose that we drop the assumption about the particular form of  $p_{ij}$  in terms of hidden parameters and make the ‘non-parametric assumption’ that the teams have a particular ordering in terms of strength that is unknown to the observer, and that *if team  $i$  is stronger than team  $j$ , then  $p_{ik} \geq p_{jk}$  for all  $k \neq i, j$* . Note that the parametric model is a special case of the non-parametric version. There are several questions: *In the non-parametric Bradley-Terry model, is it possible to estimate all the  $p_{ij}$ ’s from a tournament where every team plays against every other exactly once? Is it possible to estimate the  $p_{ij}$ ’s if only a randomly chosen fraction of the games are played? Is the estimation possible even if the fraction  $p$  tends to zero as  $n \rightarrow \infty$ ?* The Bradley-Terry model has a long history and a huge body of literature [23, 101, 40, 56, 4, 87, 85, 76, 60, 77, 41, 61, 69, 93]. It is therefore somewhat surprising that no one has considered the natural non-parametric version described above. We will show in Section 4.7 how all parameters may be simultaneously estimated in the non-parametric Bradley-Terry model.

It turns out that all of the estimation problems posed above can be solved, *without any additional assumption, by a single method*. The goal of the next section is to introduce a simple matrix estimation procedure capable of solving all of the above problems and potentially much more. The procedure will be applied to solve the posed problems in Section 4.

## 2. THE USVT ALGORITHM AND ESTIMATOR

**2.1. The setup.** Suppose that we have a  $m \times n$  matrix  $M$ , where  $m \leq n$  and the entries of  $M$  are bounded by 1 in absolute value. Let  $X$  be a matrix whose elements are independent random variables, and  $\mathbb{E}(x_{ij}) = m_{ij}$  for all  $i$  and  $j$  (where, as usual,  $x_{ij}$  and  $m_{ij}$  denote the  $(i, j)$ th entries of  $X$  and  $M$  respectively). Assume that the entries of  $X$  are also bounded by 1 in absolute value, with probability one. A matrix such as  $X$  will henceforth be called a ‘data matrix with mean  $M$ ’. The matrix  $M$  will sometimes be called the ‘parameter matrix’. Let  $p$  be a real number belonging to the interval  $[0, 1]$ . Suppose that each entry of  $X$  is observed with probability  $p$ , and unobserved with probability  $1 - p$ , independently of the other entries.

The above model will henceforth be referred to as the ‘asymmetric model’. The ‘symmetric model’ is defined in a similar manner: Take any  $n$  and let  $M$  be a symmetric matrix of order  $n$ , whose entries are bounded by 1 in absolute value. Let  $X$  be a symmetric random matrix of order  $n$  whose elements on and above the diagonal are independent, and  $\mathbb{E}(x_{ij}) = m_{ij}$  for all  $1 \leq i \leq j \leq n$ . As before, assume that the entries of  $X$  are almost surely bounded by 1 in absolute value. Take any  $p \in [0, 1]$  and suppose that each entry of  $X$  on and above the diagonal is observed with probability  $p$ , and unobserved with probability  $1 - p$ , independently of the other entries.

Similarly, one can define the ‘skew-symmetric model’, where the difference  $X - M$  is skew-symmetric, with independence on and above the diagonal as in the symmetric model. This model is used for analyzing the non-parametric Bradley-Terry model in Section 4.7.

**2.2. The USVT estimator.** In all of the above models, we construct an estimator  $\hat{M}$  of  $M$  based on the observed entries of  $X$  along the following steps. The author tentatively calls this the Universal Singular Value Thresholding (USVT) algorithm.

1. For each  $i, j$ , let  $y_{ij} = x_{ij}$  if  $x_{ij}$  is observed, and let  $y_{ij} = 0$  if  $x_{ij}$  is unobserved. Let  $Y$  be the matrix whose  $(i, j)$ th entry is  $y_{ij}$ .
2. Let  $Y = \sum_{i=1}^m s_i u_i v_i^T$  be the singular value decomposition of  $Y$ . (In the symmetric and skew-symmetric models,  $m = n$ .)
3. Let  $\hat{p}$  be the proportion of observed values of  $X$ . In the symmetric and skew-symmetric models, let  $\hat{p}$  be the proportion of observed values on and above the diagonal.
4. Let

$$S := \{i : s_i \geq 2.02\sqrt{n\hat{p}}\}.$$

[The constant 2.02 is not that important; it is simply an arbitrary choice strictly bigger than 2. The algorithm may be implemented with any constant strictly bigger than 2. Moreover, if it is known that  $\text{Var}(x_{ij}) \leq \sigma^2$  for all  $i, j$ , where  $\sigma$  is a known constant  $\leq 1$ , then the threshold  $2.02\sqrt{n\hat{p}}$  may be improved to  $2.02\sqrt{n\hat{q}}$ , where  $\hat{q} := \hat{p}\sigma^2 + \hat{p}(1 - \hat{p})(1 - \sigma^2)$ . For example, if the entries of  $X$  are known to belong to the interval  $[0, 1]$

instead of  $[-1, 1]$ , then  $\text{Var}(x_{ij}) \leq 1/4$  for all  $i, j$ , so if  $p = 1$  then in this case 2.02 may be replaced by 1.01.]

5. Define

$$W := \frac{1}{\hat{p}} \sum_{i \in S} s_i u_i v_i^T.$$

6. Let  $w_{ij}$  denote the  $(i, j)$ th element of  $W$ . Define

$$\hat{m}_{ij} := \begin{cases} w_{ij} & \text{if } -1 \leq w_{ij} \leq 1, \\ 1 & \text{if } w_{ij} > 1, \\ -1 & \text{if } w_{ij} < -1. \end{cases}$$

If the entries of  $M$  and  $X$  are known to belong to the interval  $[0, 1]$  instead of  $[-1, 1]$ , let  $\hat{m}_{ij} = 0$  when  $w_{ij} < 0$ . If  $X = M$  with probability one, let  $\hat{m}_{ij} = x_{ij}$  when  $x_{ij}$  is an observed value.

7. Let  $\hat{M}$  be the matrix whose  $(i, j)$ th entry is  $\hat{m}_{ij}$ .

**2.3. The main theorem.** Recall that the *nuclear norm* of  $M$ , written  $\|M\|_*$ , is defined as the sum of the singular values of  $M$ . Recall also the definition (1) of the mean squared error of a matrix estimator. The following theorem gives an error bound for the estimator  $\hat{M}$  in terms of the nuclear norm of  $M$ . This is the main result of this paper.

**Theorem 2.1.** *Let  $\hat{M}$  and  $M$  be as above. Let  $\text{MSE}(\hat{M})$  be defined as in (1). Suppose that  $p \geq n^{-1+\epsilon}$  for some  $\epsilon > 0$ . Then*

$$\text{MSE}(\hat{M}) \leq C \min \left\{ \frac{\|M\|_*}{m\sqrt{np}} + \frac{1}{np}, \frac{\|M\|_*^2}{mn}, 1 \right\} + C(\epsilon)e^{-cnp},$$

where  $C$  and  $c$  are positive universal constants and  $C(\epsilon)$  depends only on  $\epsilon$ . The same result holds for the symmetric and skew-symmetric models, after putting  $m = n$ . Moreover, in the case where we know that  $\text{Var}(x_{ij}) \leq \sigma^2$  for all  $i, j$  for some known  $\sigma^2 \leq 1$ , and the threshold is set at  $2.02\sqrt{n\hat{q}}$  (see Step 4 of the algorithm), the same result holds under the condition that  $q \geq n^{-1+\epsilon}$ , where  $q := p\sigma^2 + p(1-p)(1-\sigma^2)$ . In this case the exponential term in the error changes to  $C(\epsilon)e^{-cnq}$ .

Incidentally, the proof shows that the condition  $p > n^{-1+\epsilon}$  may be improved to  $p > n^{-1}(\log n)^{6+\epsilon}$  (see Theorem 8.4), but the author prefers to retain the present version for aesthetic reasons, especially considering that it is not a real improvement from any practical point of view.

It should be emphasized that although singular value thresholding has been used in a number of papers on matrix completion and estimation (e.g. [10, 1, 24, 65, 66]), the above algorithm has the unique feature that the threshold is universal. In the literature, it is usually assumed that the matrix  $M$  has a rank  $r$  that is known, and uses the value of  $r$  while thresholding. The USVT algorithm manages to cut off the singular values at the ‘correct’ level, depending on the structure of the unknown parameter matrix. The

adaptiveness of the USVT threshold is somewhat similar in spirit to that of the *SureShrink* algorithm of Donoho and Johnstone [45].

A second remark is that if  $M$  or  $X$  have entries that have absolute value greater than 1, one can easily deal with the situation by scaling  $M$  and  $X$  by an appropriate constant. Similarly, if the number of rows of  $M$  is greater than the number of columns, we work with the transposed matrices  $M^T$  and  $X^T$  instead of  $M$  and  $X$ , so that the condition  $m \leq n$  is satisfied.

**2.4. The minimax lower bound.** It is not difficult to prove that for an  $m \times n$  matrix  $M$  with entries bounded by 1 in absolute value, where  $m \leq n$ , the nuclear norm is bounded by  $m\sqrt{n}$ . Given a number  $\delta \in [0, m\sqrt{n}]$ , one may take an arbitrary estimator  $\tilde{M}$  and try to find the  $M$  among all  $M$  satisfying  $\|M\|_* \leq \delta$  for which  $\text{MSE}(\tilde{M})$  is maximum. Recall that an estimator that minimizes this maximum error is classically known as a minimax estimator. The following theorem shows that our estimator  $\hat{M}$  is minimax up to a constant multiplicative factor and an exponentially small additive discrepancy.

**Theorem 2.2.** *Consider the general matrix estimation problem outlined in the beginning of this section. Given any estimator  $\tilde{M}$  and any  $\delta \in [0, m\sqrt{n}]$ , there exists  $M$  satisfying  $\|M\|_* \leq \delta$  and a data matrix  $X$  with mean  $M$ , such that for this  $M$  and  $X$ , the estimator  $\tilde{M}$  satisfies*

$$\text{MSE}(\tilde{M}) \geq c \min \left\{ \frac{\delta}{m\sqrt{np}} + \frac{1}{np}, \frac{\delta^2}{mn}, 1 \right\},$$

where  $c$  is a positive universal constant. Moreover, if  $p < 1/2$  then  $X$  and  $M$  may be chosen such that  $X = M$ . The same lower bound holds in the symmetric case and in the skew-symmetric case.

It is worth noting that the exponentially small discrepancy is necessary. For example, if  $\delta = 0$ , then the minimax error is obviously zero. However, there is still an exponentially small chance that  $\hat{M}$  may be nonzero. It is also worth noting that if  $\delta$  is not too small (e.g. if  $\delta > \sqrt{m/p}$ ), then the exponential discrepancy does not matter, and the combination of Theorems 2.1 and 2.2 gives the correct minimax error up to a universal multiplicative constant.

**2.5. A numerical result.** Although many examples will be worked out in Section 4 and extensive simulations results will be presented in Section 7, we present the output from one set of simulations right here in this section to satisfy the reader's curiosity about the efficacy of the algorithm. The model that was simulated is a special case of the general class of latent space models discussed in Section 1. Suppose that a random graph with independent edges is defined on  $n$  vertices, where the probability that vertex  $i$  is joined to vertex  $j$  by an edge is  $m_{ij}$ , where  $m_{ij}$  obeys the logistic model

$$(2) \quad \log \frac{m_{ij}}{1 - m_{ij}} = \beta_i + \beta_j + \alpha \beta_i \beta_j.$$



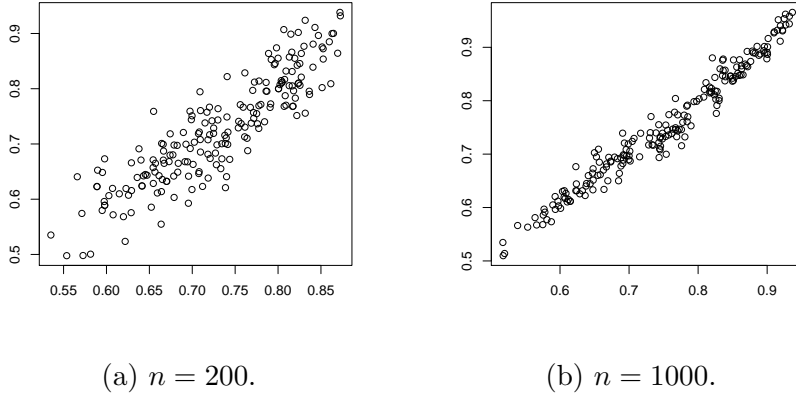


FIGURE 1. Plot of  $\hat{m}_{ij}$  vs.  $m_{ij}$  in a simulated latent space model, for a randomly chosen subset of entries of size 200 (see Section 2.5).

Here  $\alpha, \beta_1, \dots, \beta_n$  are unknown parameters. The simulations were carried out with  $n = 200$  and  $n = 1000$ , and  $\alpha, \beta_1, \dots, \beta_n$  were generated as i.i.d. Uniform $[0, 1]$  random variables. A random graph from the above model was generated, and the USVT algorithm was applied to the adjacency matrix of this random graph to estimate the  $m_{ij}$ 's. Figure 1 gives the plot of  $\hat{m}_{ij}$  versus  $m_{ij}$  as  $i, j$  vary over a subset of entries of size 200, chosen at random. (A random subset of size 200 was chosen because the set of all entries is too large to be suitable for visualization.)

Incidentally, when the interaction parameter  $\alpha$  is zero, the model is the same as the ‘beta model’ considered in [31]. In that case it is known how to compute the maximum likelihood estimates of the  $\beta_i$ 's. However when  $\alpha \neq 0$ , it is not clear how to estimate the parameters by existing methods. Figure 1 indicates that the USVT algorithm does a pretty good job even with medium sized graphs. This is quite remarkable, considering that the USVT algorithm does not ‘know’ that the graph comes from the logistic model (2); it is a fully non-parametric procedure.

This example belongs to a class of examples widely considered in this manuscript: The data is a random graph with independent edges, and the probability that there is an edge joining vertex  $i$  to vertex  $j$  is equal to  $m_{ij}$ , the  $(i, j)$ th element of a parameter matrix  $M$ . By definition,  $M$  is symmetric. Let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues of  $M$ . Theorems 2.1 and 2.2 show that  $M$  is estimable from the data if and only if  $\sum_{i=1}^n |\lambda_i|$  is small compared to  $n^{3/2}$ , and the USVT algorithm gives a way of estimating  $M$  if this condition holds. We will see in Section 4.4 that this condition is satisfied not only for the logistic model considered here, but for essentially all latent space models.

**2.6. Dependent entries.** One may wonder whether the USVT estimate works if we drop the assumption of independence. It turns out that estimation is still possible, provided we have an idea about the size of  $\|X - M\|$ , where  $\|X - M\|$  denotes the spectral norm (i.e. the largest singular value) of the matrix  $X - M$ . The modified procedure described below is never used in this article but is recorded for future purposes.

Suppose that we have an unknown matrix  $M$  of order  $m \times n$ , where  $m \leq n$ . Let  $X$  be a random matrix. Assume that all the entries of  $X$  and  $M$  are bounded by 1 in absolute value, as before. Again, as before, suppose that each entry of  $X$  is observed with probability  $p$  and unobserved with probability  $1 - p$ , independently of other entries.

Assume for simplicity that  $p$  is known. Let  $Y$  be defined as in Step 1 of the USVT algorithm. Let  $\gamma$  and  $\epsilon$  be two (known) numbers such that

$$\mathbb{P}(\|p^{-1}Y - M\| \geq \gamma) \leq \epsilon.$$

Modify the USVT algorithm as follows. Let all steps be the same, except that in Step 4, define  $S := \{i : s_i \geq 1.01\gamma\}$ . Then we have the following theorem.

**Theorem 2.3.** *In the above setting,*

$$\text{MSE}(\hat{M}) \leq \frac{C\gamma\|M\|_*}{mn} + C\epsilon,$$

where  $C$  is a universal constant.

**2.7. Plan of the paper.** Theorem 2.1 will be proved in Section 8, Theorem 2.2 in Section 9, and Theorem 2.3 in Section 10. Section 4 contains applications of Theorem 2.1 to solve the problems posed in Section 1. Section 3 compares Theorem 2.1 to the existing literature. Section 6 shows that it is impossible to estimate the mean squared error of the USVT estimate without making assumptions about the parameter matrix. Section 5 examines the situation where the data matrix  $X$  is not assumed to follow any model and discovers an interesting connection between the singular value thresholding and Szemerédi's regularity lemma from graph theory. Section 11 at the end of the manuscript contains a summary and discussion of the paper.

### 3. COMPARISON WITH THE EXISTING LITERATURE

The problem of estimating the entries of a large matrix from incomplete and/or noisy entries has received widespread attention ever since the proliferation of large data sets. Early work using spectral analysis was done by a number of authors in the computer science literature, for example by Azar et. al. [10] and Achlioptas and McSherry [1]. This was followed by a sizable body of work on spectral methods, the main pointers to which may be found in the important recent papers of Keshavan, Montanari and Oh [65, 66]. Non-spectral methods also appeared, e.g. in [88].

In a different direction, statisticians have worked on matrix completion problems under a variety of modeling assumptions. Possibly the earliest

works are due to Fazel [46] and Rudelson and Vershynin [92]. The emergence of compressed sensing [44, 27] has led to an explosion in activity in the field of matrix estimation and completion, beginning with the work of Candès and Recht [26]. The pioneering works of Emmanuel Candès and his collaborators [26, 28, 25, 24] introduced the technique of matrix completion by minimizing the nuclear norm under convex constraints, which is a convex optimization problem tractable by standard algorithms. This method has the advantage of *exactly*, rather than approximately, recovering the entries of the matrix when a suitable low rank assumption is satisfied, together with a certain other assumption called ‘incoherence’.

Since the publication of [26], a number of statistics papers have attacked the matrix completion problem from various angles. Some notable examples are [79, 81, 68, 89, 67, 38]. In a different direction, a paper that seems to have a close bearing on the analytical aspects of this paper is a manuscript of Oliveira [84].

In addition to the theoretical advances, a large number of algorithms for matrix completion and estimation have emerged. The main ones are nicely summarized and compared in [81].

The following list gives some of the key points in the comparison of this paper with the existing literature on matrix estimation and completion.

1. The literature mainly deals with low rank matrix completion and estimation. Some of the algorithms require the user to know the rank a priori (e.g. [65, 66]), whereas some others do not have this requirement (e.g. [26]). Theorem 2.1 does not require low rank, neither does it require adaptive estimation of the rank or any other quantity.
2. There are some exceptions to the low rank assumption, for example the work of Negahban and Wainwright [79] and that of Davenport et. al. [38]. The paper [38] is particularly relevant for us, since it gives the error bound in terms of the nuclear norm, just like Theorem 2.1. In fact, the authors of [38] have essentially proved the minimax rate given by the combination of Theorems 2.1 and 2.2. However, although [38] gives an error rate comparable with Theorem 2.1, the algorithm proposed in [38] is a complicated optimization problem that is sometimes (but not always) convex.
3. As mentioned above, the idea of singular value thresholding has appeared before [10, 1, 65, 66], but these methods work only for low rank matrices and require the user to know the rank of the matrix a priori. Our algorithm does not have these requirements. Incidentally, the idea of replacing missing entries by zeros has appeared earlier in [65, 66].
4. The algorithm proposed in this paper is possibly the simplest and fastest among all of the algorithms available in the literature. It involves no iterations. Only one singular value decomposition is enough. The truncation threshold does not require any adaptive estimation or any knowledge about the underlying matrix. The method is applicable for very large

matrices, where the methods based on convex optimization are possibly too slow to be useful.

5. There are no assumptions in Theorem 2.1 that are complicated or artificial. The only assumptions required are that the matrix entries are uniformly bounded and independent, and that the observed entries are chosen at random. However one should note that it may be possible to get stronger results by making more assumptions. For example, the assumption of incoherence in the papers of Candès and coauthors [26, 28] allows exact recovery of low rank matrices, whereas the minimal assumption setting of Theorem 2.1 allows only approximate recovery. On the other hand, Theorem 2.1 goes beyond low rank matrices.
6. This paper has many examples that are worked out in complete detail. This, again, diverges from the standard practice: Most papers simply indicate a list of areas and examples where the techniques may be applicable. However, there are some exceptions such as [79].

#### 4. EXAMPLES

Throughout this section,  $m$ ,  $n$ ,  $M$ ,  $X$ ,  $p$  and  $\hat{M}$  will be as in Section 2. Just to remind the reader,  $M$  is an  $m \times n$  matrix where  $1 \leq m \leq n$ . The entries of  $M$  are assumed to be bounded by 1 in absolute value. The matrix  $X$  is a random matrix whose entries are independent, and the  $(i, j)$ th element  $x_{ij}$  has expected value equal to  $m_{ij}$ , the  $(i, j)$ th entry of  $M$ . Moreover, they satisfy  $|x_{ij}| \leq 1$  with probability one. In particular,  $X$  may be exactly equal to  $M$ , with no randomness. Each entry of  $X$  is observed with probability  $p$  and unobserved with probability  $1 - p$ , independently of other entries. Occasionally, we will assume the symmetric model, where  $m = n$ , and the matrices  $M$  and  $X$  are symmetric. In the special case of the Bradley-Terry model in Section 4.7, we will assume the skew-symmetric model, where  $X - M$  is skew-symmetric.

We will now work out various specific cases where Theorem 2.1 gives useful results.

**4.1. Low rank matrices.** As mentioned in Section 3, estimating low rank matrices has been the focus of the vast majority of prior work [10, 1, 46, 88, 92, 26, 25, 28, 24, 79, 65, 66, 68, 67, 81]. Theorem 2.1 works for low rank matrices. The following theorem, which is a simple corollary of Theorem 2.1, shows that  $\hat{M}$  is a good estimate whenever the rank of  $M$  is small compared to  $mp$  (after assuming, as in Theorem 2.1, that  $p \geq n^{-1+\epsilon}$ ).

**Theorem 4.1.** *Suppose that  $M$  has rank  $r$ . Suppose that  $p \geq n^{-1+\epsilon}$  for some  $\epsilon > 0$ . Then*

$$\text{MSE}(\hat{M}) \leq C \min \left\{ \sqrt{\frac{r}{mp}} + \frac{1}{np}, 1 \right\} + C(\epsilon)e^{-cnp}.$$

where  $C$  and  $c$  are universal constants and  $C(\epsilon)$  depends only on  $\epsilon$ . Moreover, the same result holds when  $M$  and  $X$  are symmetric.

*Proof.* Recall that the Frobenius norm of  $M$  is defined as

$$\|M\|_F := (\text{Tr}(M^T M))^{1/2} = \left( \sum_{i=1}^m \sum_{j=1}^n m_{ij}^2 \right)^{1/2}.$$

Note that  $\|M\|_F^2$  is also the sum of squares of the singular values of  $M$ . The number of nonzero singular values is equal to the rank of  $M$ . Therefore by the Cauchy-Schwarz inequality,

$$(3) \quad \|M\|_* \leq \sqrt{\text{rank}(M)} \|M\|_F,$$

and therefore  $\|M\|_* \leq \sqrt{r m n}$ . The result now follows from Theorem 2.1.  $\square$

The term  $1/n p$  in the error bound is necessary to take care of the case  $r = 0$ . Even if  $M$  is identically zero, the estimator  $\hat{M}$  will incur some error due to the (possible) randomness in  $X$ .

Let us now inspect how the condition  $r \ll m p$  compares with available bounds. Keshavan, Montanari and Oh [65, 66] obtain the same condition, but only if  $m$  and  $n$  are comparable. Theorem 4.1, on the other hand, works for ‘very rectangular’ matrices also, where  $m \ll n$ . Moreover, the approach of Keshavan et. al. requires the user to know the rank a priori.

Candès and Tao [28] obtain the condition  $r \ll m p$  with an extra poly-logarithmic term in the error. Moreover they too require that  $m$  and  $n$  be comparable, and additionally they need the so-called ‘incoherence condition’. However, as noted before, the incoherence condition allows exact recovery, while our approach only gives approximate recovery.

The recent work of Davenport et. al. [38] gives an estimator with an error bound that is almost the same as that given by Theorem 4.1, but with a complicated optimization algorithm.

The following theorem shows that the condition  $r \ll m p$  is necessary for estimating  $M$ .

**Theorem 4.2.** *Given any estimator  $\tilde{M}$ , there exists an  $m \times n$  matrix  $M$  of rank  $r$  with entries bounded between  $-1$  and  $1$ , such that when the data is sampled from  $M$ ,*

$$\text{MSE}(\tilde{M}) \geq C(1 - p)^{\lfloor m/r \rfloor},$$

where  $C$  is a positive universal constant and  $\lfloor m/r \rfloor$  is the integer part of  $m/r$ .

*Proof.* Let  $M$  be an  $m \times n$  random matrix whose first  $r$  rows consist of i.i.d. Uniform $[-1, 1]$  random variables, and copy this block  $\lfloor m/r \rfloor$  times. Declare the remaining rows, if any, to be zero. Then note that  $M$  has rank  $\leq r$ .

Let  $D$  be our data, that is, the observed values of  $M$ . One can imagine  $D$  as a matrix whose  $(i, j)$ th entry is  $m_{ij}$  if  $m_{ij}$  is observed, and a question mark if  $m_{ij}$  is unobserved. For any  $(i, j)$  belonging to the nonzero portion of the matrix  $M$ ,  $M$  contains  $\lfloor m/r \rfloor$  copies of  $m_{ij}$ . Since the  $M$ -value at the location of each copy is observed with probability  $p$ , independent of the other copies, the chance that none of these copies are observed is equal to  $(1 - p)^{\lfloor m/r \rfloor}$ . If none of the copies are observed, then the data contains no

information about  $m_{ij}$ . Using this, it is not difficult to write down a formal argument that shows

$$\mathbb{E}(\text{Var}(m_{ij} \mid D)) \geq C(1-p)^{\lceil m/r \rceil}$$

where  $C$  is some universal constant. On the other hand, since  $\tilde{m}_{ij}$  is a function of  $D$ , the definition of variance implies that

$$\mathbb{E}((\tilde{m}_{ij} - m_{ij})^2 \mid D) \geq \text{Var}(m_{ij} \mid D).$$

Combining the last two displays, we see that

$$\mathbb{E}\|\tilde{M} - M\|_F^2 \geq Cmn(1-p)^{\lceil m/r \rceil}.$$

This completes the proof.  $\square$

**4.2. The stochastic blockmodel.** Consider an undirected graph on  $n$  vertices. A stochastic blockmodel assumes that the vertices  $1, \dots, n$  are partitioned into  $k$  blocks, and the probability that vertex  $i$  is connected to vertex  $j$  by an edge depends only on the blocks to which  $i$  and  $j$  belong. As usual, edges are independent of each other. Let  $M$  be the matrix whose  $(i, j)$ th element is the probability of an edge existing between vertices  $i$  and  $j$ . The matrix  $X$  here is the adjacency matrix of the observed graph. Here all elements of  $X$  are observed, so  $p = 1$ .

This is commonly known as the stochastic blockmodel. It was introduced by Holland, Laskey and Leinhardt [58] as a simple stochastic model of social networks. It has become one of the most successful and widely used models for community structure in networks, especially after the advent of large data sets.

Early analysis of the stochastic blockmodel was carried out by Snijders and Nowicki [96, 82], who provided consistent parameter estimates when there are exactly two blocks. This was extended to a finite but fixed number of blocks of equal size by Condon and Karp [37]. Bickel and Chen [16] were the first to give consistent estimates for finite number of blocks of unequal size. It was observed by Leskovec et. al. [71] that in real data, the number of blocks often seem to grow with the number of nodes. This situation was rigorously analyzed for the first time in Rohe et. al. [90], and was followed up shortly thereafter by [17, 36, 78, 34] with more advanced results.

However, all in all, the author is not aware of any estimator for the stochastic blockmodel that works whenever the number of blocks is small compared to the number of nodes. The best result till date is in the very recent manuscript of Rohe et. al. [91], who prove that a penalized likelihood estimator works whenever  $k$  is comparable to  $n$  ‘up to log factors’. The following theorem says that the USVT estimator  $\hat{M}$  gives a complete solution to the estimation problem in the stochastic blockmodel if  $k \ll n$ , with no further conditions required. (The method will not work very well for sparse graphs, however; for recent advances on estimation in sparse graphs, see [35].)

**Theorem 4.3.** *For a stochastic blockmodel with  $k$  blocks,*

$$\text{MSE}(\hat{M}) \leq C \sqrt{\frac{k}{n}},$$

where  $C$  is a universal constant.

*Proof.* If two vertices  $i$  and  $j$  are in the same block, then the  $i$ th and  $j$ th rows of  $M$  are identical. Therefore  $M$  has at most  $k$  distinct rows and so the rank of  $M$  is  $\leq k$ . An application of Theorem 4.1 completes the proof.  $\square$

Note that estimating the stochastic blockmodel is a special case of low rank matrix estimation with noise. It is not difficult to prove that the estimation problem is impossible when  $k$  is of the same order as  $n$ . We will not bother to write down a formal proof.

**4.3. Distance matrices.** Suppose that  $K$  is a compact metric space with metric  $d$ . Let  $x_1, \dots, x_n$  be arbitrary points from  $K$ , and let  $M$  be the  $n \times n$  matrix whose  $(i, j)$ th entry is  $d(x_i, x_j)$ . Such matrices are called ‘distance matrices’. Since  $K$  is a compact metric space, the diameter of  $K$  with respect to the metric  $d$  must be finite. Scaling  $d$  by a constant factor, we may assume without loss of generality that the diameter is bounded by 1, so that the entries of  $M$  are bounded by 1 as required by Theorem 2.1.

Completing a distance matrix with missing entries has been a popular problem in the engineering and social sciences for a long time; see, for example, [97, 11, 7, 18, 94, 95]. It has become particularly relevant in engineering problems related to sensor networks. It is also an important issue in multi-dimensional scaling [19]. For some recent theoretical advances, see [83, 62].

In general, distance matrices need not be of low rank. Therefore much of the literature on matrix estimation and completion does not apply to distance matrices. Surprisingly, Theorem 2.1 gives a complete solution of the distance matrix completion and estimation problem.

**Theorem 4.4.** *Suppose that  $p \geq n^{-1+\epsilon}$  for some  $\epsilon > 0$ . If  $M$  is a distance matrix as above, then*

$$\text{MSE}(\hat{M}) \leq \frac{C(K, d, n)}{\sqrt{p}} + C(\epsilon)e^{-cnp},$$

where  $c$  is a universal constant,  $C(\epsilon)$  depends only on  $\epsilon$ , and  $C(K, d, n)$  is number depending only on  $K$ ,  $d$  and  $n$  such that

$$\lim_{n \rightarrow \infty} C(K, d, n) = 0.$$

The above theorem is not wholly satisfactory, since it does not indicate how fast  $p$  can go to zero as  $n \rightarrow \infty$  so that  $\hat{M}$  is still consistent. To understand that, we need to know more about the structure of the space  $K$ . The following theorem gives a quantitative estimate.

**Theorem 4.5.** *Suppose that for each  $\delta > 0$ ,  $N(\delta)$  is a number such that  $K$  may be covered by  $N(\delta)$  open  $d$ -balls of radius  $\delta$ . Then*

$$\text{MSE}(\hat{M}) \leq C \inf_{\delta > 0} \min \left\{ \frac{\delta + \sqrt{N(\delta/4)/n}}{\sqrt{p}}, 1 \right\} + C(\epsilon)e^{-cnp},$$

where  $C$  and  $c$  are universal constants and  $C(\epsilon)$  depends only on  $\epsilon$ .

To see how Theorem 4.5 may be used, suppose that  $K$  is a compact subset of the real line and  $d$  is the usual distance on  $\mathbb{R}$ , scaled by a factor to ensure that the diameter of  $K$  is  $\leq 1$ . Then  $N(\delta)$  increases like  $1/\delta$  as  $\delta \rightarrow 0$ . Consequently, given  $n$ , the optimal choice of  $\delta$  is of the order  $n^{-1/3}$ , which gives the bound

$$\text{MSE}(\hat{M}) \leq \frac{Cn^{-1/3}}{\sqrt{p}}.$$

(Note that the exponential term need not appear because the main term is bounded below by a positive universal constant if  $p < n^{-2/3}$ .) Thus,  $\hat{M}$  is a consistent estimate as long as  $p$  goes to zero slower than  $n^{-2/3}$  as  $n \rightarrow \infty$ .

The proofs of Theorems 4.4 and 4.5 follow from a more general lemma that will also be useful later for other purposes. Suppose that  $S = \{x_1, \dots, x_n\}$  is a finite set and  $f : S \times S \rightarrow [-1, 1]$  is an arbitrary function. Suppose that for each  $\delta > 0$ , there exists a partition  $\mathcal{P}(\delta)$  of  $S$  such that whenever  $x, y, x', y'$  are four points in  $S$  such that  $x, x' \in P$  for some  $P \in \mathcal{P}(\delta)$  and  $y, y' \in Q$  for some  $Q \in \mathcal{P}(\delta)$ , then  $|f(x, y) - f(x', y')| \leq \delta$ . Let  $M$  be the  $n \times n$  matrix whose  $(i, j)$ th element is  $f(x_i, x_j)$ .

**Lemma 4.6.** *In the above setting,*

$$\text{MSE}(\hat{M}) \leq C \inf_{\delta > 0} \min \left\{ \frac{\delta + \sqrt{|\mathcal{P}(\delta)|/n}}{\sqrt{p}}, 1 \right\} + C(\epsilon)e^{-cnp},$$

where  $C$  and  $c$  are universal constants, and  $C(\epsilon)$  depends only on  $\epsilon$ .

*Proof.* Fix some  $\delta > 0$ . Let  $T$  be a subset of  $S$  consisting of exactly one point from each member of  $\mathcal{P}(\delta)$ . For each  $x \in S$ , let  $p(x)$  be the unique element of  $T$  such that  $x$  and  $p(x)$  belong to the same element of  $\mathcal{P}(\delta)$ . Let  $N$  be the matrix whose  $(i, j)$ th element is  $f(p(x_i), p(x_j))$ . Then

$$\|M - N\|_F^2 = \sum_{i,j=1}^n (f(x_i, x_j) - f(p(x_i), p(x_j)))^2 \leq n^2 \delta^2.$$

By the triangle inequality for the nuclear norm, the inequality (3) and the above inequality,

$$\begin{aligned} \|M\|_* &\leq \|M - N\|_* + \|N\|_* \\ &\leq \sqrt{n} \|M - N\|_F + \|N\|_* \\ &\leq n^{3/2} \delta + \|N\|_*. \end{aligned}$$



Now, if  $x_i$  and  $x_j$  belong to the same element of  $\mathcal{P}(\delta)$ , then  $p(x_i) = p(x_j)$  and hence the  $i$ th and  $j$ th rows of  $N$  are identical. This shows that  $N$  has at most  $|\mathcal{P}(\delta)|$  distinct rows and therefore has rank  $\leq |\mathcal{P}(\delta)|$ . Therefore by the inequality (3),

$$\|N\|_* \leq \sqrt{|\mathcal{P}(\delta)|} \|N\|_F \leq \sqrt{|\mathcal{P}(\delta)|} n.$$

The proof is completed by applying Theorem 2.1.  $\square$

Using Lemma 4.6, it is easy to prove Theorems 4.4 and 4.5.

*Proof of Theorem 4.5.* Let all notation be as in Theorem 4.5. To apply Lemma 4.6, let  $S$  be the set  $\{x_1, \dots, x_n\}$ . From the definition of  $N(\delta)$ , it is easy to see that there is a partition  $\mathcal{P}(\delta)$  of  $S$  of size  $\leq N(\delta/4)$ , such that any two points belonging to the same element of the partition are at distance  $\leq \delta/2$  from each other. Consequently, if  $x, x' \in P$  and  $y, y' \in Q$  for some  $P, Q \in \mathcal{P}(\delta)$ , then by the triangle inequality for the metric  $d$ ,

$$\begin{aligned} |d(x, y) - d(x', y')| &\leq |d(x, y) - d(x', y)| + |d(x', y) - d(x', y')| \\ &\leq d(x, x') + d(y, y') \leq \delta. \end{aligned}$$

Putting  $f = d$  in Lemma 4.6, the proof is complete.  $\square$

*Proof of Theorem 4.4.* Since  $K$  is compact, there exists a finite number  $N(\delta)$  for each  $\delta > 0$  such that  $K$  may be covered by  $N(\delta)$  open  $d$ -balls of radius  $\delta$ . By Theorem 4.5, this shows that for any sequence  $\delta_n$  decreasing to zero,

$$\text{MSE}(\hat{M}) \leq C \min \left\{ \frac{\delta_n + \sqrt{N(\delta_n/4)/n}}{\sqrt{p}}, 1 \right\} + C(\epsilon)e^{-cnp}.$$

To complete the proof, choose  $\delta_n$  going to zero so slowly that  $N(\delta_n/4) = o(n)$  as  $n \rightarrow \infty$ .  $\square$

**4.4. Latent space models.** Suppose that  $\beta_1, \dots, \beta_n$  are vectors belonging to some bounded closed set  $K \subseteq \mathbb{R}^k$ , where  $k$  is some arbitrary but fixed dimension. Let  $f : K \rightarrow [-1, 1]$  be a continuous function. Let  $M$  be the  $n \times n$  matrix whose  $(i, j)$ th element is  $f(\beta_i, \beta_j)$ . Then our data matrix  $X$  has the form

$$x_{ij} = f(\beta_i, \beta_j) + \epsilon_{ij},$$

where  $\epsilon_{ij}$  are independent errors with zero mean, satisfying the restriction that  $|x_{ij}| \leq 1$  almost surely. For example,  $X$  may be the adjacency matrix of a random graph where the probability of an edge existing between vertices  $i$  and  $j$  is  $f(\beta_i, \beta_j)$ . This is one context where latent space models are widely used, starting with the work of Hoff, Raftery and Handcock [57]. A large body of work applying the latent space approach to real data has grown in the last decade. On the theoretical side, it was observed in [16, 17] that the latent space model arises naturally from an exchangeability assumption due to the Aldous-Hoover theorem [6, 59]. Note that distance matrices and stochastic blockmodels are both special cases of latent space models.

There have been various attempts to estimate parameters in the latent space models (e.g. [57, 55, 5]). Almost all of these approaches rely on heuristic arguments and justification through simulations, without any kind of proof whatsoever. The problem is that in addition to the vectors  $\beta_1, \dots, \beta_n$ , the function  $f$  itself is an unknown parameter. If either  $\beta_i$ 's are known, or  $f$  is known, the estimation problem is tractable. For example, when  $f(x, y)$  is of the form  $e^{x+y}/(1 + e^{x+y})$ , the problem was solved in [31]. However when both  $f$  and  $\beta_i$ 's are unknown, the problem becomes seemingly intractable. In particular, there is an identifiability issue because  $f(x, y)$  may be replaced by  $h(x, y) := f(g(x), g(y))$  and  $\beta_i$  by  $g^{-1}(\beta_i)$  for any invertible function  $g$  without altering the model.

In view of the above discussion, it is a rather surprising consequence of Theorem 2.1 that it is possible to estimate the numbers  $f(\beta_i, \beta_j)$ ,  $i, j = 1, \dots, n$  from a single realization of the data matrix, under no additional assumptions than the stated ones.

**Theorem 4.7.** *Suppose that  $p \geq n^{-1+\epsilon}$ . If  $M$  is as above, then*

$$\text{MSE}(\hat{M}) \leq \frac{C(K, k, f, n)}{\sqrt{p}} + C(\epsilon)e^{-cnp},$$

where  $c$  is a universal constant,  $C(\epsilon)$  depends only on  $\epsilon$ , and  $C(K, k, f, n)$  depends only on  $K, k, f$  and  $n$  such that

$$\lim_{n \rightarrow \infty} C(K, k, f, n) = 0.$$

*Proof.* We will apply Lemma 4.6. Let  $S$  be the set  $\{\beta_1, \dots, \beta_n\}$ . Since  $f$  is continuous on  $K$  and  $K$  is compact,  $f$  must be uniformly continuous. This shows that for each  $\delta > 0$  we can find a partition  $\mathcal{P}(\delta)$  of  $S$  satisfying the condition required for Lemma 4.6, such that the size of  $\mathcal{P}(\delta)$  may be bounded by a constant  $N(\delta)$  depending only on  $K, k, f$  and  $\delta$ . Choosing  $\delta_n \rightarrow 0$  slowly enough so that  $N(\delta_n/4) = o(n)$  and applying Lemma 4.6 completes the proof.  $\square$

The problem with Theorem 4.7, just like Theorem 4.4 in Section 4.3, is that it does not give an explicit error bound, which makes it impossible to determine how fast  $p$  can go to zero with  $n$  so that consistency holds. Again, this is easy to fix by assuming smoothness properties of  $f$  and applying Lemma 4.6. As a particular example, suppose that  $f$  is Lipschitz with Lipschitz constant  $L$ , in the sense that

$$|f(x, y) - f(x', y')| \leq L\|x - x'\| + L\|y - y'\|$$

for all  $x, y, x', y' \in K$ .

**Theorem 4.8.** *In the above setting,*

$$\text{MSE}(\hat{M}) \leq C(K, k, L) \frac{n^{-1/(k+2)}}{\sqrt{p}},$$

where  $C(K, k, L)$  is a constant depending only on  $K, k$  and  $L$ .

*Proof.* Let  $S = \{\beta_1, \dots, \beta_n\}$ . Take any  $\delta > 0$ . From the Lipschitzness condition, it is easy to see that we can find a partition  $\mathcal{P}(\delta)$  of  $S$  whose size may be bounded by  $C(K, k, L)\delta^{-k}$ , where  $C(K, k, L)$  depends only on  $K$ ,  $k$  and  $L$ . Choosing  $\delta = n^{-1/(k+2)}$  and applying Lemma 4.6 completes the proof. Note that the exponential term need not appear since the main term is bounded below by a positive universal constant if  $p < n^{-2/(k+2)}$ .  $\square$

**4.5. Positive definite matrices.** Assume that  $m = n$  and  $M$  is positive semidefinite. (In the statistical context, this is the same as saying that  $M$  is a covariance matrix. When the diagonal entries are all 1,  $M$  is a correlation matrix.)

Completing positive definite matrices with missing entries has received a lot of attention in the linear algebra literature [53, 63, 15], although most of the techniques are applicable only for relatively small matrices or when a sizable fraction of the entries are observed. In the engineering sciences, estimation of covariance matrices from a small subset of observed entries arises in the field of remote sensing (see [25, 26, 28] for brief discussions).

The statistical matrix completion literature cited in Section 3 applies only to low rank positive definite matrices. It is therefore quite a surprise that the completion problem may be solved for *any* positive definite matrix whenever we get to observe a large number of entries from each row.

**Theorem 4.9.** *Suppose that  $m = n$  and  $M$  is positive semidefinite. Suppose that  $p \geq n^{-1+\epsilon}$ . Then*

$$\text{MSE}(\hat{M}) \leq \frac{C}{\sqrt{np}} + C(\epsilon)e^{-cnp},$$

where  $C$  and  $c$  are universal constants and  $C(\epsilon)$  depends only on  $\epsilon$ .

*Proof.* Since  $M$  is positive semidefinite,  $\|M\|_* = \text{Tr}(M)$ . Since the entries of  $M$  are bounded by 1,  $\text{Tr}(M) \leq n$ . The proof now follows from an application of Theorem 2.1.  $\square$

What if  $p$  is of order  $1/n$  or less? The following theorem shows that it is impossible to estimate  $M$  in this situation.

**Theorem 4.10.** *Given any estimator  $\tilde{M}$ , there exists a correlation matrix  $M$  such that when the data is sampled from  $M$ ,*

$$\text{MSE}(\tilde{M}) \geq C(1-p)^n,$$

where  $C$  is a positive universal constant.

*Proof.* Throughout this proof,  $C$  will denote any positive universal constant, whose value may change from line to line.

Let  $U_1, \dots, U_n$  be i.i.d. Uniform $[0, 1]$  random variables. Let  $M$  be the random matrix whose  $(i, j)$ th element  $m_{ij}$  is equal to  $U_i U_j$  if  $i \neq j$  and 1 if  $i = j$ . It is easy to verify that  $M$  is a correlation matrix. Suppose that we observe each element of  $M$  on and above the diagonal with probability

$p$ , independent of each other. Let  $D$  be our data, represented as follows:  $D$  is a matrix whose  $(i, j)$ th element is  $m_{ij}$  if the element is observed, and a question mark otherwise.

Now, the probability that no element from the  $i$ th row and the  $i$ th column is observed is exactly equal to  $(1 - p)^n$ . If we do not observe any element from the  $i$ th row and  $i$ th column, we have no information about the value of  $U_i$ . From this, it is not difficult to write down a formal argument to prove that for any  $j \neq i$ ,

$$\text{Var}(m_{ij} \mid D, (U_k)_{k \neq i}) = U_j^2 \text{Var}(U_i \mid D, (U_k)_{k \neq i}) \geq C(1 - p)^n U_j^2.$$

If  $\tilde{M}$  is any estimator, then  $\tilde{m}_{ij}$  is a function of  $D$ . Therefore by the above inequality and the definition of variance,

$$\mathbb{E}((\tilde{m}_{ij} - m_{ij})^2 \mid D, (U_k)_{k \neq i}) \geq C(1 - p)^n U_j^2,$$

and thus

$$\mathbb{E}(\tilde{m}_{ij} - m_{ij})^2 \geq C(1 - p)^n.$$

Since this is true for all  $i \neq j$ , the proof is complete.  $\square$

**4.6. Graphons.** A graphon is a measurable function  $f : [0, 1]^2 \rightarrow [0, 1]$  that satisfies  $f(x, y) \equiv f(y, x)$ . The term ‘graphon’ was coined by Lovász and coauthors in the growing literature on limits of dense graphs [20, 21, 22, 72, 73, 74]. Such functions also arise in the related study of weakly exchangeable random arrays [42, 9, 6, 59]. They have also appeared recently in large deviations [32, 33, 75] and mathematical statistics [30, 86].

In the graph limits literature, graphons arise as limits of graphs with increasing number of nodes. Conversely, graphons are often used to generate random graphs in a natural way. Take any  $n$  and let  $U_1, \dots, U_n$  be i.i.d. Uniform $[0, 1]$  random variables. Construct a random undirected graph on  $n$  vertices by putting an edge between vertices  $i$  and  $j$  with probability  $f(U_i, U_j)$ , doing this independently for all  $1 \leq i < j \leq n$ . This procedure is sometimes called ‘sampling from a graphon’ (see [21, Section 4.4]).

The statistical question is the following: Suppose that we have a random graph on  $n$  vertices that is sampled from a graphon. Is it possible to estimate the graphon from a single realization of the graph? More precisely, is it possible to accurately estimate the numbers  $f(U_i, U_j)$ ,  $1 \leq i < j \leq n$ , from a single realization of the random graph? The question is similar to the one investigated in Section 4.3, but the difference is that here we are not allowed to assume any regularity on  $f$  except measurability.

Taking things back to our usual setting, let  $M$  be the matrix whose  $(i, j)$ th element is  $f(U_i, U_j)$ . Note that unlike our previous examples,  $M$  is now random. So the definition of MSE should be modified to take expectation over  $M$  as well.

**Theorem 4.11.** *In the above setting,*

$$\text{MSE}(\hat{M}) \leq C(f, n),$$

where  $C(f, n)$  is a constant depending only on  $f$  and  $n$ , such that

$$\lim_{n \rightarrow \infty} C(f, n) = 0.$$

*Proof.* Here all entries of the adjacency matrix are visible, so  $p = 1$ . Define a sequence of functions  $f_1, f_2, \dots$  according the following standard construction. For each  $k$ , let  $\mathcal{P}_k$  be the  $k$ th level dyadic partition of  $[0, 1)^2$ , that is, the partition of the unit square into sets of the form  $[(i-1)/2^k, i/2^k) \times [(j-1)/2^k, j/2^k)$ . Let  $f_k$  be the function that is equal to the average value of  $f$  within each square of the partition  $\mathcal{P}_k$ . If  $\mathcal{F}_k$  denotes the sigma-algebra of sets generated by the partition  $\mathcal{P}_k$ , then the sequence  $f_k$  is a martingale with respect to the filtration  $\mathcal{F}_k$ . Moreover,  $f_k = \mathbb{E}(f|\mathcal{F}_k)$ . Finally, observe that the sequence  $f_k$  is uniformly bounded in  $L^2$ . Combining all these observations, it is evident that  $f_k \rightarrow f$  in  $L^2$ .

Now fix some  $\epsilon > 0$  and an integer  $n$ . Take a large enough  $k = k(\epsilon)$  such that  $\|f - f_k\|_{L^2} \leq \epsilon$ . Let  $N$  be the  $n \times n$  matrix whose  $(i, j)$ th element is  $f_k(U_i, U_j)$ . Then

$$\begin{aligned} \mathbb{E}\|M - N\|_F^2 &= \sum_{i,j=1}^n \mathbb{E}(f(U_i, U_j) - f_k(U_i, U_j))^2 \\ &\leq n + n^2 \mathbb{E}(f(U_1, U_2) - f_k(U_1, U_2))^2 \\ (4) \quad &= n + n^2 \|f - f_k\|_{L^2}^2 \leq n + n^2 \epsilon^2. \end{aligned}$$

Now note that if  $U_i$  and  $U_j$  belong to the same dyadic interval  $[r/2^k, (r+1)/2^k)$ , then the  $i$ th and  $j$ th rows of  $N$  are identical. Hence  $N$  has at most  $2^k$  distinct rows, and therefore has rank  $\leq 2^k$ . Therefore by (3),

$$\|N\|_* \leq 2^{k/2} \|N\|_F \leq 2^{k/2} n.$$

Consequently, by the inequality (3),

$$\begin{aligned} \|M\|_* &\leq \|M - N\|_* + \|N\|_* \\ (5) \quad &\leq \sqrt{n} \|M - N\|_F + 2^{k/2} n. \end{aligned}$$

Combining (4) and (5) gives

$$\mathbb{E}\|M\|_* \leq (2^{k/2} + 1)n + n^{3/2}\epsilon.$$

Choosing a sequence  $\epsilon_n$  going to zero so slowly that  $2^{k(\epsilon_n)/2} = o(n^{-1/2})$ , we can now apply Theorem 2.1 to complete the proof.  $\square$

**4.7. Non-parametric Bradley-Terry model.** Suppose there are  $n$  teams playing against each other in a tournament. Every team plays against every other team at least once (often, exactly once). Suppose that  $p_{ij}$  is the probability that team  $i$  wins against team  $j$  in a match between  $i$  and  $j$ . Then  $p_{ji} = 1 - p_{ij}$ .

The Bradley-Terry model [23], originally proposed by Zermelo [101], assumes that  $p_{ij}$  is of the form  $a_i/(a_i + a_j)$  for some unknown nonnegative numbers  $a_1, \dots, a_n$ . It is known how to estimate the parameters  $a_1, \dots, a_n$

if we assume that the outcomes of all games are independent — which, in this case, is a reasonable assumption.

The Bradley-Terry model has found great success among practitioners. For an old survey of the literature on the model dating back to 1976, see [40]. Numerous extensions and applications have been proposed, for example [56, 4, 87, 85, 76, 77, 60]. The monographs of David [39] and Diaconis [41, Chapter 9] explain the statistical foundations of these models. More recently, several authors have proposed to perform Bayesian inference for (generalized) Bradley-Terry models [3, 50, 51, 52, 54, 29].

For the basic Bradley-Terry model, it is possible to find the maximum likelihood estimate of the  $a_i$ 's using a simple iterative procedure [101, 61, 69]. The maximum likelihood estimate was shown to be jointly consistent for all  $n$  parameters by Simons and Yao [93].

We now generalize the Bradley-Terry model as follows. Suppose, as before, that  $p_{ij}$  is the probability that team  $i$  beats team  $j$ . Suppose that the teams have a particular ordering in terms of strength that is unknown to the observer. Assume that *if team  $i$  is stronger than team  $j$ , then  $p_{ik} \geq p_{jk}$  for all  $k \neq i, j$* . Do not assume anything else about the  $p_{ij}$ 's; in particular, do not assume any formula for the  $p_{ij}$ 's in terms of hidden parameters. This is what we may call a ‘non-parametric Bradley-Terry model’. Note that the usual Bradley-Terry model is a special case of the non-parametric version.

In the non-parametric Bradley-Terry model, is it possible to estimate all the  $p_{ij}$ 's from a tournament where every team plays against every other exactly once? Is it possible to estimate the  $p_{ij}$ 's if only a randomly chosen fraction of the games are played? How small can this fraction be, so that accurate estimation is still possible? The following theorem provides some answers.

**Theorem 4.12.** *Consider the non-parametric Bradley-Terry model defined above. Let  $M$  be the matrix whose  $(i, j)$ th entry is  $p_{ij}$  if  $i \neq j$  and 0 if  $i = j$ . Let  $X$  be the data matrix whose  $(i, j)$ th entry is 1 if team  $i$  won over team  $j$ , 0 if team  $j$  won over team  $i$  and recorded as missing if team  $i$  did not play versus team  $j$ . If team  $i$  has played against team  $j$  multiple times, let the  $(i, j)$ th entry of  $X$  be the proportion of times that  $i$  won over  $j$ . (Draws are not allowed.) Let all diagonal entries of  $X$  be zero. Given  $p \in [0, 1]$ , suppose that for each  $i$  and  $j$ , the game between  $i$  and  $j$  takes place with probability  $p$  and does not take place with probability  $1 - p$ , independent of other games. Let  $\hat{M}$  be the estimate of  $M$  based on the data matrix  $X$ . Then*

$$\text{MSE}(\hat{M}) \leq \frac{Cn^{-1/3}}{\sqrt{p}},$$

where  $C$  is a universal constant. In particular, the estimation problem is solvable whenever  $p \gg n^{-2/3}$ .

*Proof.* As usual, throughout the proof  $C$  will denote any universal constant, whose value may change from line to line.

Recall that the definition of the skew-symmetric model stipulates that  $X - M$  is skew-symmetric, which is true for the non-parametric Bradley-Terry model. There is nothing to prove if  $p < n^{-2/3}$ , so assume that  $p \geq n^{-2/3}$ . This allows us to drop the exponential term in Theorem 2.1 and conclude that

$$(6) \quad \text{MSE}(\hat{M}) \leq \frac{C\|M\|_*}{n^{3/2}\sqrt{p}} + \frac{C}{np}.$$

Let  $k$  be an integer less than  $n$ , to be determined later. For each  $i$ , let

$$t_i := \sum_{j=1}^n p_{ij}.$$

Note that each  $t_i$  belongs to the interval  $[0, n]$ . For  $l = 1, \dots, k$ , let  $T_l$  be the set of all  $i$  such that  $t_i \in [n(l-1)/k, nl/k)$ . Additionally, if  $t_i = n$ , put  $i$  in  $T_k$ .

For each  $l$ , let  $r(l)$  be a distinguished element of  $T_l$ . For each  $1 \leq i, j \leq n$ , if  $i \in T_l$  and  $j \in T_m$ , let  $n_{ij} := p_{r(l)j}$ . Let  $N$  be the matrix whose  $(i, j)$ th element is  $n_{ij}$ . Note that if  $i, i' \in T_l$  for some  $l$ , then  $n_{ij} = n_{i'j}$  for all  $j$ . In particular,  $N$  has at most  $k$  distinct rows and therefore has rank  $\leq k$ . Thus by inequality (3),

$$(7) \quad \|N\|_* \leq \sqrt{k}\|N\|_F \leq n\sqrt{k}.$$

Now take any  $1 \leq i \leq n$ . Suppose that  $i \in T_l$ . Let  $i' = r(l)$ . Suppose that team  $i'$  is weaker than team  $i$ . Then  $p_{i'j} \leq p_{ij}$  for all  $j \neq i, i'$ . Thus,

$$\begin{aligned} \sum_{j=1}^n (p_{ij} - n_{ij})^2 &= \sum_{j=1}^n (p_{ij} - p_{i'j})^2 \leq \sum_{j=1}^n |p_{ij} - p_{i'j}| \\ &= \sum_{j \neq i, i'} (p_{ij} - p_{i'j}) + p_{ii'} + p_{i'i} \\ (8) \quad &\leq t_i - t_{i'} + 2 \leq \frac{n}{k} + 2 \leq \frac{3n}{k}. \end{aligned}$$

Similarly, if team  $i'$  is stronger than team  $i$ ,

$$(9) \quad \sum_{j=1}^n (p_{ij} - n_{ij})^2 \leq t_{i'} - t_i + 2 \leq \frac{3n}{k}.$$

By (7), (8), (9) and (3) we have

$$\begin{aligned} \|M\|_* &\leq \|M - N\|_* + \|N\|_* \\ &\leq \sqrt{n}\|M - N\|_F + n\sqrt{k} \\ &\leq \frac{3n^{3/2}}{k} + n\sqrt{k}. \end{aligned}$$

Choosing  $k = \lceil n^{1/3} \rceil$ , we get

$$\|M\|_* \leq Cn^{7/6}.$$

Together with (6), this shows that

$$\text{MSE}(\hat{M}) \leq \frac{Cn^{-1/3}}{\sqrt{p}} + \frac{C}{np}.$$

Since  $p \geq n^{-2/3}$ , the second term on the right is redundant.  $\square$

A natural question is whether the threshold  $p \gg n^{-2/3}$  is sharp. The author does not know the answer, but suspects that the estimation problem is solvable all the way up to  $p \gg n^{-1}$ .

**4.8. Structured matrices.** This section will try to justify the claim that the USVT procedure works for any matrix that has a ‘little bit of structure’. For this one needs to define a notion of structure in a matrix. Our definition, informally, is that a square matrix is ‘structured’ if upon a suitable permutation of rows and columns, it resembles a measurable function on the unit square. (For now, we will stick to square matrices only. It should not be difficult to generalize to rectangular matrices.)

Let  $\mathcal{M}$  be the space of functions  $f : [0, 1]^2 \rightarrow \mathbb{R}$  such that  $|f(x, y)| \leq 1$  for all  $x$  and  $y$ . Since the elements of  $\mathcal{M}$  are uniformly bounded,  $L^p$  norms on  $\mathcal{M}$  are equivalent for all  $p \in (0, \infty)$ . We will henceforth consider the  $L^2$  topology as the natural topology on  $\mathcal{M}$ .

A square matrix  $M$  of order  $n$  may be canonically represented as an element of  $\mathcal{M}$  in the following way. Given  $M = (m_{ij})_{1 \leq i, j \leq n}$ , define the function  $f^M$  on  $[0, 1]^2$  as

$$f^M(x, y) = m_{ij} \quad \text{if} \quad \frac{i-1}{n} < x \leq \frac{i}{n} \quad \text{and} \quad \frac{j-1}{n} < y \leq \frac{j}{n},$$

and if either  $x = 0$  or  $y = 0$ , let  $f^M(x, y) = 0$ . With the above convention, notice that for any two square matrices  $M, N$  of order  $n$ ,

$$(10) \quad \|f^M - f^N\|_{L^2}^2 = \frac{1}{n^2} \sum_{i,j=1}^n (m_{ij} - n_{ij})^2 = \frac{1}{n^2} \text{Tr}[(M - N)^2].$$

Given a matrix  $M$  and permutations  $\pi, \sigma \in S_n$  (where  $S_n$  is the group of all permutations of  $\{1, \dots, n\}$ ), let  $\pi M \sigma$  be the matrix whose  $(i, j)$ th element is  $m_{\pi(i)\sigma(j)}$ .

**Theorem 4.13.** *Suppose that we have a sequence of parameter matrices  $M_n$ , where  $M_n$  is of order  $n$  and symmetric, two sequences of permutations  $\pi_n$  and  $\sigma_n$ , where  $\pi_n, \sigma_n \in S_n$ , and a measurable function  $f : [0, 1]^2 \rightarrow \mathbb{R}$ , such that*

$$f^{\pi_n M_n \sigma_n} \rightarrow f \quad \text{in } L^2 \text{ as } n \rightarrow \infty.$$

*For each  $n$ , let  $X_n$  be a data matrix with mean  $M_n$  and assume all the usual conditions for the symmetric model. For each  $n$ , let  $p_n$  be a number such that each entry on and above the diagonal of  $X_n$  is observed with probability  $p_n$  and unobserved with probability  $1 - p_n$ , independently of other entries.*



Let  $\hat{M}_n$  be the USVT estimate of  $M_n$  computed using the observed entries of  $X_n$ . Suppose that for all  $n$ ,  $p_n \geq n^{-1+\epsilon}$  for some fixed  $\epsilon > 0$ . Then

$$\text{MSE}(\hat{M}_n) \leq \frac{C_n}{\sqrt{p_n}} + C(\epsilon)e^{-cnp_n},$$

where  $c$  is a universal constant,  $C(\epsilon)$  depends only on  $\epsilon$ , and  $C_n$  is a sequence of constants that do not depend on  $p_n$  and satisfy  $\lim_{n \rightarrow \infty} C_n = 0$ . In particular, if  $p_n$  stays bounded away from zero (or goes to zero sufficiently slowly), then

$$\lim_{n \rightarrow \infty} \text{MSE}(\hat{M}_n) = 0.$$

*Proof.* By Theorem 2.1,

$$\text{MSE}(\hat{M}_n) \leq \frac{C\|M_n\|_*}{n^{3/2}\sqrt{p_n}} + C(\epsilon)e^{-cnp_n},$$

where  $C$  and  $c$  are universal constants, and  $C(\epsilon)$  depends only on  $\epsilon$ . Therefore it suffices to show that

$$\lim_{n \rightarrow \infty} \frac{\|M_n\|_*}{n^{3/2}} = 0.$$

Since  $\|M_n\|_* = \|\pi_n M_n \sigma_n\|_*$  and the USVT procedure is equivariant under permutations of rows and columns, it suffices to prove the theorem under the assumption that  $f^{M_n} \rightarrow f$  in  $L^2$ .

Take any two positive integers  $n$  and  $m$ . Let  $M_{n,m} := M_n \otimes J_m$ , where  $J_m$  is the  $m \times m$  matrix of all 1's and  $\otimes$  denotes the Kronecker product. In other words,  $M_{n,m}$  is an  $n \times n$  array of square blocks of order  $m$ , with all elements of the  $(i, j)$ th block equal to the  $(i, j)$ th element of  $M_n$ . Similarly, define  $M_{m,n} := M_m \otimes J_n$ . Note that, as functions on  $[0, 1]^2$ ,  $f^{M_n} = f^{M_{n,m}}$  and  $f^{M_m} = f^{M_{m,n}}$ . Thus,

$$(11) \quad \|f^{M_{n,m}} - f^{M_{m,n}}\|_{L^2} = \|f^{M_n} - f^{M_m}\|_{L^2}.$$

Let  $\lambda_1, \dots, \lambda_n$  be the singular values of  $M_n$ , arranged in decreasing order. The singular values of  $M_{n,m}$  are precisely  $m\lambda_1, m\lambda_2, \dots, m\lambda_n$  and  $mn - n$  zeroes. Similarly, if  $\delta_1, \dots, \delta_m$  are the singular values of  $M_m$ , then the singular values of  $M_{m,n}$  are  $n\delta_1, \dots, n\delta_m$  and  $mn - m$  zeroes.

Create a vector of length  $mn$  by arranging the singular values of  $M_{n,m}$  in decreasing order. Do the same for the singular values of  $M_{m,n}$ . Pair each element of the first vector with the corresponding element of the second. For  $1 \leq i \leq n$ , let  $\mu_i$  be the singular value of  $M_{m,n}$  that is paired with the singular value  $m\lambda_i$  of  $M_{n,m}$ . Let  $\mu_{n+1}, \dots, \mu_{mn}$  be the remaining  $mn - n$  singular values of  $M_{m,n}$ . Then by a well-known result from linear algebra (see Theorem 8.1 in Section 8.2) and the identities (11) and (10),

$$(12) \quad \begin{aligned} \sum_{i=1}^n (m\lambda_i - \mu_i)^2 + \sum_{i=n+1}^{mn} \mu_i^2 &\leq \text{Tr}[(M_{n,m} - M_{m,n})^2] \\ &= m^2 n^2 \|f^{M_n} - f^{M_m}\|_{L^2}^2. \end{aligned}$$

Now note that

$$\begin{aligned}
 n \sum_{i=1}^m \delta_i &= \sum_{i=1}^{mn} \mu_i = \sum_{i=1}^n \mu_i + \sum_{i=n+1}^{mn} \mu_i \\
 (13) \quad &\leq m \sum_{i=1}^n \lambda_i + \sum_{i=1}^n |m\lambda_i - \mu_i| + \sum_{i=n+1}^{mn} \mu_i.
 \end{aligned}$$

By the Cauchy-Schwarz inequality and (12),

$$\begin{aligned}
 \sum_{i=1}^n |m\lambda_i - \mu_i| &\leq \left( n \sum_{i=1}^n (m\lambda_i - \mu_i)^2 \right)^{1/2} \\
 (14) \quad &\leq mn^{3/2} \|f^{M_n} - f^{M_m}\|_{L^2}.
 \end{aligned}$$

Again, since there are at most  $m$  nonzero  $\mu_i$ 's, another application of the Cauchy-Schwarz inequality and (12) gives

$$\begin{aligned}
 \sum_{i=n+1}^{mn} \mu_i &\leq \left( m \sum_{i=n+1}^{mn} \mu_i^2 \right)^{1/2} \\
 (15) \quad &\leq m^{3/2} n \|f^{M_m} - f^{M_n}\|_{L^2}.
 \end{aligned}$$

Combining (13), (14) and (15), we get

$$n \sum_{i=1}^m \delta_i \leq m \sum_{i=1}^n \lambda_i + (mn^{3/2} + m^{3/2}n) \|f^{M_n} - f^{M_m}\|_{L^2}.$$

Dividing both sides by  $m^{3/2}n$  gives

$$\frac{\|M_m\|_*}{m^{3/2}} \leq \frac{\|M_n\|_*}{n\sqrt{m}} + (n^{1/2}m^{-1/2} + 1) \|f^{M_n} - f^{M_m}\|_{L^2}.$$

Now let  $m \rightarrow \infty$ , keeping  $n$  fixed. This gives

$$\limsup_{m \rightarrow \infty} \frac{\|M_m\|_*}{m^{3/2}} \leq \|f^{M_n} - f\|_{L^2}.$$

The proof is completed by taking  $n \rightarrow \infty$  on the right.  $\square$

## 5. THE MODEL-FREE CASE: A CONNECTION WITH SZEMERÉDI'S LEMMA

Let  $X$  be an  $m \times n$  matrix (where  $m \leq n$ ) whose entries belong to the interval  $[-1, 1]$ . Let  $X = \sum_{i=1}^m s_i u_i v_i^T$  be the singular value decomposition of  $X$ . Choose some  $\kappa > 0$  and define

$$X^\kappa := \sum_{i: s_i > \kappa n} s_i u_i v_i^T.$$

In the USVT algorithm for the case of no missing entries,  $\kappa = 2.02 n^{-1/2}$ .

The question is, what does  $X^\kappa$  represent when  $X$  is an arbitrary matrix, not occurring as a realization of some randomized model? Roughly speaking, the answer is as follows: Declare two rows of  $X$  to be 'similar' if the two corresponding rows of  $X^\kappa$  are close to each other in some reasonable metric.

Similarly, call two columns of  $X$  ‘similar’ if the two corresponding columns of  $X^\kappa$  are close to each other in some reasonable metric. It turns out that if  $U$  is a bunch of rows of  $X$  that are similar to each other, and if  $V$  is a bunch of columns of  $X$  that are similar to each other, then the submatrix  $(x_{ij})_{i \in U, j \in V}$  is ‘approximately random with independent entries’. To put it more succinctly,  *$X$  looks like a random matrix from the stochastic blockmodel ensemble, with blocks defined using the matrix  $X^\kappa$ .*

For a precise understanding of the above paragraph, one needs to understand the setting of a famous result of Szemerédi, known as the ‘regularity lemma’. The result may be described as follows. Let  $G = (V, E)$  be a simple graph, and let  $Y, Z$  be subsets of  $V$ . Then we denote by  $e(Y, Z)$  the number of  $Y$ - $Z$  edges of  $G$ , and define

$$\rho(Y, Z) := \frac{e(Y, Z)}{|Y||Z|}.$$

Here  $|Y|$  denotes the size of  $Y$ . Given some  $\epsilon > 0$ , call a pair  $(A, B)$  of disjoint sets  $A, B \subseteq V$   $\epsilon$ -regular if all  $Y \subseteq A$  and  $Z \subseteq B$  with  $|Y| \geq \epsilon|A|$  and  $|Z| \geq \epsilon|B|$  satisfy

$$|\rho(Y, Z) - \rho(A, B)| \leq \epsilon.$$

A partition  $\{V_0, \dots, V_K\}$  of  $V$  is called an  $\epsilon$ -regular partition of  $G$  if it satisfies the following three conditions:

- (i)  $|V_0| \leq \epsilon n$ ;
- (ii)  $|V_1| = |V_2| = \dots = |V_K|$ ;
- (iii) all but at most  $\epsilon K^2$  of the pairs  $(V_i, V_j)$  with  $1 \leq i < j \leq K$  are  $\epsilon$ -regular.

Szemerédi’s regularity lemma goes as follows.

**Theorem 5.1** (Szemerédi’s lemma [98]). *Given  $\epsilon > 0$  and an integer  $m \geq 1$  there exists an integer  $M = M(\epsilon, m)$  such that every graph of order at least  $M$  admits an  $\epsilon$ -regular partition  $\{V_0, \dots, V_K\}$  for some  $K$  in the range  $m \leq K \leq M$ .*

This result was proved by Szemerédi [98] in 1976 and has found many applications in combinatorics, number theory and other areas of discrete mathematics. The version presented above is from Diestel [43, Section 7.2]. We will not use this result anywhere in this manuscript; it is stated here simply to put the discussion in perspective.

The definition of an  $\epsilon$ -regular pair is of relevance to us. When a pair of subsets  $(A, B)$  is  $\epsilon$ -regular, the intuitive meaning is that the edges between  $A$  and  $B$  are ‘approximately randomly distributed’. The smaller  $\epsilon$  is, the more random is the distribution of edges. Alternatively, one may say that the edges are ‘uniformly’ or ‘regularly’ distributed.

One may similarly define a notion of  $\epsilon$ -regularity for submatrices of our data matrix  $X$ . For any  $U \subseteq \{1, \dots, m\}$  and  $V \subseteq \{1, \dots, n\}$ , define

$$\rho(U, V) := \frac{\sum_{i \in U, j \in V} x_{ij}}{|U||V|},$$

and say that the pair  $(U, V)$  is  $\epsilon$ -regular if for any  $U' \subseteq U$  and  $V' \subseteq V$  with  $|U'| \geq \epsilon|U|$  and  $|V'| \geq \epsilon|V|$ , we have

$$|\rho(U', V') - \rho(U, V)| \leq \epsilon.$$

Having defined regularity, let us now define the notion of ‘similarity’ of rows and columns that was alluded to in the beginning of this section. Let  $x_{ij}^\kappa$  denote the  $(i, j)$ th entry of  $X^\kappa$ . Define a distance  $r^\kappa(i, i')$  between the  $i$ th and  $i'$ th rows of  $X$  as

$$r^\kappa(i, i') := \frac{1}{n} \sum_{j=1}^n |x_{ij}^\kappa - x_{i'j}^\kappa|.$$

Similarly, define a distance  $c^\kappa(j, j')$  between two columns  $j$  and  $j'$ . For a subset  $U$  of  $\{1, \dots, m\}$ , define

$$r^\kappa(U) := \max_{i, i' \in U} r^\kappa(i, i').$$

Similarly, for  $V \subseteq \{1, \dots, n\}$  define

$$c^\kappa(V) := \max_{j, j' \in V} c^\kappa(j, j').$$

Let  $\kappa$  be small. If  $U$  is a set of rows such that  $r^\kappa(U)$  is small, then the rows in  $U$  are ‘similar’ to each other. The notion of ‘similar columns’ is defined in the same manner using  $c^\kappa$ . Note that similarity is defined using  $X^\kappa$ , whereas the notion of regularity refers to the original data matrix  $X$ .

The following theorem shows that if  $U$  is a bunch of similar rows and  $V$  is a bunch of similar columns, and if the sizes of  $U$  and  $V$  are not too small, then the pair  $(U, V)$  is  $\epsilon$ -regular for some small  $\epsilon$ . The result may be useful for clustering rows and columns in real-life applications.

**Theorem 5.2.** *Take any three numbers  $\kappa$ ,  $\delta$  and  $\tau$  in the interval  $(0, 1]$ . Let  $\alpha := \sqrt{n/m}$ . Let  $U$  be a subset of  $\{1, \dots, m\}$  and  $V$  be a subset of  $\{1, \dots, n\}$  such that  $r^\kappa(U) \leq \delta$  and  $c^\kappa(V) \leq \delta$ . Additionally, suppose that  $|U| \geq \tau m$  and  $|V| \geq \tau n$ . Then the pair  $(U, V)$  is  $\epsilon$ -regular, where  $\epsilon = \sqrt{2(\kappa\alpha + \delta)}/\tau$ .*

*Proof.* Define  $\rho^\kappa(U, V)$  the same way that  $\rho(U, V)$  was defined, but using  $x_{ij}^\kappa$  instead of  $x_{ij}$ . Take any  $U' \subseteq U$  and  $V' \subseteq V$  such that  $|U'| \geq \epsilon|U|$  and

$|V'| \geq \epsilon|V|$ . Then

$$\begin{aligned}
 |\rho(U', V') - \rho^\kappa(U', V')| &= \frac{|\sum_{i \in U', j \in V'} (x_{ij} - x_{ij}^\kappa)|}{|U'| |V'|} \\
 &\leq \frac{\|X - X^\kappa\|}{\sqrt{|U'| |V'|}} \leq \frac{\kappa n}{\sqrt{|U'| |V'|}} \\
 (16) \qquad &\leq \frac{\kappa n}{\epsilon \sqrt{|U| |V|}} \leq \frac{\kappa \alpha}{\epsilon \tau}.
 \end{aligned}$$

Similarly,

$$(17) \qquad |\rho(U, V) - \rho^\kappa(U, V)| \leq \frac{\|X - X^\kappa\|}{\sqrt{|U| |V|}} \leq \frac{\kappa n}{\sqrt{|U| |V|}} \leq \frac{\kappa \alpha}{\tau}.$$

Let  $U''$  be any subset of  $U$  of the same size as  $U'$ . Then

$$\begin{aligned}
 |\rho^\kappa(U', V') - \rho^\kappa(U'', V')| &= \left| \frac{\sum_{i \in U', j \in V'} x_{ij}^\kappa}{|U'| |V'|} - \frac{\sum_{i \in U'', j \in V'} x_{ij}^\kappa}{|U''| |V'|} \right| \\
 &\leq \frac{1}{|U'| |V'|} |U'| n r^\kappa(U) \leq \frac{\delta}{\epsilon \tau}.
 \end{aligned}$$

Similarly, if  $V''$  is a subset of  $V$  of the same size as  $V$ , then

$$|\rho^\kappa(U'', V') - \rho^\kappa(U'', V'')| \leq \frac{\delta}{\epsilon \tau}.$$

Combining the last two displays we see that for any  $U'' \subseteq U$  and  $V'' \subseteq V$  such that  $|U''| = |U'|$  and  $|V''| = |V'|$ ,

$$|\rho^\kappa(U', V') - \rho^\kappa(U'', V'')| \leq \frac{2\delta}{\epsilon \tau}.$$

Averaging this inequality over all such  $U''$  and  $V''$  and applying Jensen's inequality, we get

$$|\rho^\kappa(U', V') - \rho^\kappa(U, V)| \leq \frac{2\delta}{\epsilon \tau}.$$

Combining this with (16) and (17) gives

$$|\rho(U', V') - \rho(U, V)| \leq \frac{2(\kappa \alpha + \delta)}{\epsilon \tau} = \epsilon.$$

This completes the proof.  $\square$

## 6. AN IMPOSSIBILITY THEOREM FOR ERROR ESTIMATES

Theorem 2.1 gives an upper bound on the mean squared error of  $\hat{M}$ . The estimate involves the nuclear norm of parameter matrix  $M$ . A natural question is: Is it possible to estimate the true MSE of  $\hat{M}$  from the data?

A straightforward approach is to use parametric bootstrap. Having estimated  $M$  using  $\hat{M}$ , one may choose a large number  $K$ , generate  $K$  copies of the data using  $\hat{M}$  as the parameter matrix, compute the estimates  $\hat{M}^{(i)}$ ,

$i = 1, \dots, K$  for the  $K$  simulations, and estimate the MSE of  $\hat{M}$  using the bootstrap estimator

$$\widehat{\text{MSE}}_{\text{BS}}(\hat{M}) = \frac{1}{K} \sum_{i=1}^K \frac{\|\hat{M}^{(i)} - \hat{M}\|_F^2}{mn}.$$

For the validity of the bootstrap estimate of the MSE, it is essential that the original  $\hat{M}$  is an accurate estimate of  $M$ . In other words, we need to know a priori that  $\text{MSE}(\hat{M})$  is small to be able to claim that the bootstrap estimator of  $\text{MSE}(\hat{M})$  is accurate. Theorem 2.1 implies that if we know that  $\|M\|_*$  is small enough from assumptions, this is true. For example, in all of the examples from Section 4, it is plausible that the bootstrap method may be used to estimate the MSE of  $\hat{M}$  (although the author does not quite know how to prove that).

But is it possible to somehow determine whether  $\text{MSE}(\hat{M})$  is small or not from the data, if we do not make any assumption about  $M$  to start with? We will now show that it is impossible to do so, not only for the estimator  $\hat{M}$  but for any ‘non-trivial’ estimator  $\tilde{M}$ .

The definition of a non-trivial estimator is as follows. Given a parameter matrix  $M$  and a data matrix  $X$  satisfying the conditions of Section 2, the trivial estimator of  $M$  based on  $X$  is simply  $X$  itself. We will denote the trivial estimator as  $\hat{M}^{\text{Trv}}$ . Now suppose we are given some estimator  $\tilde{M}$ . We will say that the estimator  $\tilde{M}$  is non-trivial if there exists a sequence of parameter matrices  $M_n$  and data matrices  $X_n$  such that

$$\text{MSE}(\hat{M}_n^{\text{Trv}}) \not\rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

but  $\lim_{n \rightarrow \infty} \text{MSE}(\tilde{M}_n) = 0$ . In other words,  $\tilde{M}_n$  solves a non-trivial estimation problem. The USVT estimator is clearly non-trivial, as demonstrated by the examples from Section 4.

Suppose that we have a non-trivial estimator  $\tilde{M}$  and a procedure  $P$  that gives an estimate  $\widehat{\text{MSE}}_P(\tilde{M})$  of the MSE of  $\tilde{M}$ . The MSE estimate is computed using only the data. The procedure will be called ‘good’ if the following two conditions hold:

- (a) Whenever  $M_n$  is a sequence of parameter matrices and  $X_n$  is a sequence of data matrices such that  $\text{MSE}(\tilde{M}_n)$  tends to zero, the estimate  $\widehat{\text{MSE}}_P(\tilde{M}_n)$  also tends to zero in probability.
- (b) Whenever  $M_n$  is a sequence of parameter matrices and  $X_n$  is a sequence of data matrices such that  $\text{MSE}(\tilde{M}_n)$  does not tend to zero,  $\widehat{\text{MSE}}_P(\tilde{M}_n)$  also does not tend to zero in probability.

**Theorem 6.1.** *There cannot exist a good procedure for estimating the mean squared error of a non-trivial estimator.*

*Proof.* Suppose that a good procedure  $P$  exists. By the definition of non-triviality of the estimator  $\tilde{M}$ , there exists a sequence of parameter matrices

$M_n$  and data matrices  $X_n$  such that

$$(18) \quad \text{MSE}(\hat{M}_n^{\text{Trv}}) \not\rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

but

$$(19) \quad \lim_{n \rightarrow \infty} \text{MSE}(\tilde{M}_n) = 0.$$

Then by the definition of goodness,

$$\widehat{\text{MSE}}_{\text{P}}(\tilde{M}_n) \rightarrow 0 \quad \text{in probability as } n \rightarrow \infty.$$

Suppose, without loss of generality, that all the data matrices are defined on the same probability space. Then taking a subsequence if necessary, we may assume that in addition to (18) and (19), we also have

$$(20) \quad \mathbb{P}\left(\lim_{n \rightarrow \infty} \widehat{\text{MSE}}_{\text{P}}(\tilde{M}_n) = 0\right) = 1.$$

Let  $M'_n := X_n$  and  $X'_n := X_n$  for all  $n$ . Consider  $M'_n$  as a (random) parameter matrix and  $X'_n$  as its data matrix. Given  $M'_n$ , the expected value of  $X'_n$  is  $M'_n$ ; so it is okay to treat  $M'_n$  as a parameter matrix and  $X'_n$  as its data matrix. We will denote the estimate of  $M'_n$  constructed using  $X'_n$  as  $\tilde{M}'_n$ . Note that since  $M'_n$  is random, the mean squared error of  $\tilde{M}'_n$  is a random variable.

Now, since the estimator  $\tilde{M}'_n$  is computed using the data matrix only, and  $X'_n = X_n$ , it is clear that  $\tilde{M}'_n = \tilde{M}_n$ . There is no randomness in  $\tilde{M}'_n$  when  $M'_n$  is given, since  $X'_n = M'_n$ . Thus, if  $r_n$  and  $c_n$  denote the number of rows and columns of  $M_n$ , then

$$\begin{aligned} \text{MSE}(\tilde{M}'_n) &= \frac{1}{r_n c_n} \|\tilde{M}'_n - M'_n\|_F^2 \\ &= \frac{1}{r_n c_n} \|\tilde{M}_n - X_n\|_F^2 \\ &= \frac{1}{r_n c_n} \|\tilde{M}_n - \hat{M}_n^{\text{Trv}}\|_F^2 \\ &\geq \frac{1}{2r_n c_n} \|\hat{M}_n^{\text{Trv}} - M_n\|_F^2 - \frac{1}{r_n c_n} \|\tilde{M}_n - M_n\|_F^2, \end{aligned}$$

where the last step follows from the inequality  $(a + b)^2 \leq 2a^2 + 2b^2$  and the triangle inequality for the Frobenius norm. Taking expectation on both sides gives

$$\mathbb{E}(\text{MSE}(\tilde{M}'_n)) \geq \frac{1}{2} \text{MSE}(\hat{M}_n^{\text{Trv}}) - \text{MSE}(\tilde{M}_n).$$

Therefore by (18) and (19),

$$\mathbb{E}(\text{MSE}(\tilde{M}'_n)) \not\rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In particular, since mean squared errors are uniformly bounded by 1,

$$(21) \quad \mathbb{P}(\text{MSE}(\tilde{M}'_n) \not\rightarrow 0 \quad \text{as } n \rightarrow \infty) > 0.$$

Again since  $\widehat{\text{MSE}}_{\text{P}}$  is computed using the data matrix only, therefore for all  $n$ ,

$$\widehat{\text{MSE}}_{\text{P}}(\tilde{M}_n) = \widehat{\text{MSE}}_{\text{P}}(\tilde{M}'_n).$$

Therefore by (20),

$$(22) \quad \mathbb{P}\left(\lim_{n \rightarrow \infty} \widehat{\text{MSE}}_{\text{P}}(\tilde{M}'_n) = 0\right) = 1.$$

The equations (21) and (22) demonstrate the existence of a sequence of parameter matrices  $M'_n$  and data matrices  $X'_n$  such that  $\text{MSE}(\tilde{M}'_n) \not\rightarrow 0$  but  $\widehat{\text{MSE}}_{\text{P}}(\tilde{M}'_n) \rightarrow 0$  in probability. This contradicts the goodness of  $\widehat{\text{MSE}}_{\text{P}}$ .  $\square$

## 7. SIMULATION RESULTS

Although the USVT seems like a useful algorithm in theory, the unspecified constant in Theorem 2.1 might make it a little suspicious in the eyes of a practitioner. To lay such fears to rest, the author resorted to applying the USVT algorithm to simulated data and computing the root mean squared errors (RMSE) of the obtained estimates, defined as

$$\text{RMSE}(\hat{M}) = \left[ \mathbb{E} \left( \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (\hat{m}_{ij} - m_{ij})^2 \right) \right]^{1/2}.$$

Actually, what is reported below is not the RMSE, but the following estimate of the RMSE:

$$\widehat{\text{RMSE}}(\hat{M}) := \left( \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (\hat{m}_{ij} - m_{ij})^2 \right)^{1/2}.$$

It was observed in simulations that  $\widehat{\text{RMSE}}$  has fairly small fluctuations for large matrices, and therefore may be used as a surrogate for the true RMSE. Abusing terminology, we will sometimes refer to  $\widehat{\text{RMSE}}$  as RMSE.

**7.1. Blockmodels.** The first batch of simulations investigated applications of the USVT algorithm to the stochastic blockmodel. Given two numbers  $n$  and  $k$ , each node among  $1, \dots, n$  was assigned randomly to one of  $k$  blocks. A random symmetric  $k \times k$  matrix  $P = (p_{ab})_{1 \leq a, b \leq k}$  was generated, whose elements on and above the diagonal were chosen to be i.i.d. Uniform $[0, 1]$  random variables. Given  $P$ , an  $n \times n$  matrix  $M$  was defined as follows. If a node  $i$  belongs to block  $a$  and another node  $j$  belongs to block  $b$ , let  $m_{ij} = p_{ab}$ . If  $i = j$ , let  $m_{ij} = 0$ . Given the matrix  $M$ , a graph  $G$  on  $n$  vertices was generated to putting an edge between vertex  $i$  and vertex  $j$  with probability  $m_{ij}$ , doing this independently for all  $1 \leq i < j \leq n$ . The adjacency matrix  $X$  of  $G$  was taken as the data matrix, and the estimate  $\hat{M}$  and its root mean squared error were computed. The results for various values of  $n$  and  $k$  are reported below.



$n$	$k = 2$	$k = 5$	$k = 10$
100	0.065	0.139	0.165
200	0.070	0.087	0.141
500	0.050	0.066	0.082
1000	0.033	0.045	0.060

Table 1: Estimated RMSE in blockmodel simulation

**7.2. Latent space models.** The second batch of simulations was done for latent space models. Alternatively, this may also be considered as simulations for graphons. A function  $f : [0, 1]^2 \rightarrow [0, 1]$  was chosen, and  $\beta_1, \dots, \beta_n$  were generated as i.i.d. uniform random variables from the interval  $[0, 1]$ . The  $(i, j)$ th entry of the matrix  $M$  was set to be  $m_{ij} = f(\beta_i, \beta_j)$ . Data was generated by choosing  $x_{ij}$  to be a 0-1 valued random variable with  $\mathbb{P}(x_{ij} = 1) = m_{ij}$ . The root mean squared errors for various choices of  $f$  and  $n$  are reported below. The first two functions are examples of low rank models. The third one is a positive definite functions of full rank. The fourth one is a logistic model that appears in [31]; it is similar to the Bradley-Terry model [23].

$f(x, y)$	$n = 100$	$n = 200$	$n = 1000$
$\frac{1}{2}(x + y)$	0.095	0.073	0.035
$xy$	0.075	0.053	0.027
$\min\{x, y\}$	0.097	0.075	0.038
$1/(1 + e^{-x-y})$	0.098	0.069	0.032

Table 2: Estimated RMSE in latent space model simulation

**7.3. Covariance matrix completion.** The third batch of simulations was done for covariance matrix completion. A bunch of number  $x_1, \dots, x_n$  were chosen as i.i.d. uniform random variables from the interval  $[0, 1]$ , and the  $(i, j)$ th entry of  $M$  was defined as  $m_{ij} = \min\{x_i, x_j\}$ . It is not difficult to verify that this is a positive definite matrix. This particular matrix was chosen because it is a positive definite matrix that is not of low rank;  $M$  has rank  $n$  with probability 1. A fraction  $p$  of randomly chosen elements of  $M$  was used as data for estimating  $M$ . The estimated RMSE for various values of  $n$  and  $p$  are reported below. They seem to be comparable with the upper bound of  $1/\sqrt{np}$  given by Theorem 4.9.

$n$	$p = 0.5$	$p = 0.2$	$p = 0.1$
100	0.067	0.350	0.442
200	0.048	0.096	0.393
1000	0.038	0.059	0.079

Table 3: Estimated RMSE in latent space model simulation

**7.4. Distance matrix completion.** A fourth batch of simulations was done for distance matrices. Just for variety,  $n$  numbers  $x_1, \dots, x_n$  were chosen independently and uniformly at random from the unit cube in  $\mathbb{R}^3$ . The  $(i, j)$ th entry of  $M$  was set to be  $m_{ij} = d(x_i, x_j)$  where  $d$  is the Euclidean distance divided by  $\sqrt{3}$  (so that all entries are in  $[0, 1]$ ). A fraction  $p$  of randomly chosen entries of  $M$  were observed. The estimated RMSE for various values of  $n$  and  $p$  are reported below. From the simulations, it does seem that a larger sample size is needed for accuracy in dimension three, as indicated by the upper bound in Theorem 4.5.

$n$	$p = 0.5$	$p = 0.2$	$p = 0.1$
100	0.102	0.365	0.388
200	0.094	0.133	0.377
1000	0.090	0.117	0.131
2000	0.028	0.115	0.125

Table 4: Estimated RMSE in Euclidean distance matrix simulation

Euclidean distance matrices are of low rank if the entries are the squared distances instead of actual distances. Therefore one may argue that the distance matrix completion problem may be solved one of the low rank algorithms. However, if one considers the  $\ell^1$  distance instead of Euclidean distance, the matrix of distances (or any power of it) is no longer of low rank. Simulation results are presented below. Note that here the  $\ell^1$  distance was divided by 3, so that all distances are  $\leq 1$ .

$n$	$p = 0.5$	$p = 0.2$	$p = 0.1$
100	0.091	0.318	0.337
200	0.094	0.123	0.334
1000	0.087	0.114	0.125
2000	0.030	0.113	0.121

Table 5: Estimated RMSE in  $\ell^1$  distance matrix simulation

**7.5. Collaborative filtering.** In recommender systems, vendors provide recommendations based on ratings submitted by users. Typically, each user rates only a few items. A special instance of this problem is the famous Netflix problem [2]. Users (columns of the data matrix) rate movies (rows of the data matrix) on a scale of 1 to 5. The number of available ratings is small compared to the number of ratings that are missing. Completing this matrix would enable the vendor to recommend movies to users. This is an example of a general class of problems known as collaborative filtering [49]. A fifth and final batch of simulations was done for a simulated Netflix-type problem. In the simulated model, there were 500 movies and 5000 users. Each movie belonged to one of  $k$  genres and each user belonged to one of  $l$  types. A bunch of numbers  $p_{ab}$ ,  $a = 1, \dots, k$ ,  $b = 1, \dots, l$  were chosen independently and uniformly at random from the set  $\{1, 2, 3, 4, 5\}$ . For each movie  $i$  of genre  $a$  and each user  $j$  of type  $b$  we assumed that  $j$  gives rating  $p_{ab}$  to movie  $i$ . Only a fraction  $p$  of the ratings, randomly chosen, were observed. The estimated RMSE for various values of  $k$ ,  $l$  and  $p$  are reported below.

$k, l$	$p = 0.5$	$p = 0.2$	$p = 0.1$
2, 3	0.154	0.481	0.502
4, 10	0.466	1.129	1.226
10, 30	0.949	1.208	1.315

Table 6: Estimated RMSE in Netflix-type simulation

**7.6. Further work.** The simulations in this section and in Section 2.5 provide some evidence of the efficacy of the USVT method in practice. More extensive simulations, applications to real data, and numerical comparisons with existing techniques have been planned for a forthcoming project.

## 8. PROOF OF THEOREM 2.1

**8.1. Matrix norms.** Let  $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$  be an  $m \times n$  real matrix with singular values  $\sigma_1, \dots, \sigma_k$ , where  $k = \min\{m, n\}$ . The following matrix norms are widely used in this proof.

The nuclear norm or the trace norm of  $A$  is defined as

$$\|A\|_* := \sum_{i=1}^k \sigma_i.$$

The Frobenius norm, also called the Hilbert-Schmidt norm, is defined as

$$\|A\|_F := \left( \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \right)^{1/2} = (\text{Tr}(A^T A))^{1/2} = \left( \sum_{i=1}^k \sigma_i^2 \right)^{1/2}.$$

Recall that, as noted in the inequality (3),

$$\|A\|_* \leq \sqrt{\text{rank}(A)} \|A\|_F.$$

The sup-norm is defined as

$$\|A\|_\infty := \max_{i,j} |a_{ij}|.$$

The spectral norm or the operator norm of  $A$  is defined as

$$\|A\| := \max_{1 \leq i \leq k} |\sigma_i|.$$

The spectral norm may be alternatively expressed as

$$(23) \quad \|A\| = \max_{x \in \mathbb{S}^{m-1}, y \in \mathbb{S}^{n-1}} x^T A y,$$

where  $\mathbb{S}^{m-1}$  and  $\mathbb{S}^{n-1}$  are the Euclidean unit spheres in  $\mathbb{R}^m$  and  $\mathbb{R}^n$  respectively. The above representation implies that the spectral norm satisfies the triangle inequality. Consequently, for any two  $m \times n$  matrices  $A$  and  $B$ ,

$$|\|A\| - \|B\|| \leq \|A - B\| \leq \|A - B\|_F.$$

In particular, the spectral norm is a Lipschitz function of the matrix entries (with Lipschitz constant 1), if the entries are collectively considered as a vector of length  $mn$ .

The triangle inequality for the spectral norm also implies that the map  $A \mapsto \|A\|$  is convex. Indeed, for any  $0 \leq t \leq 1$ ,

$$\|tA + (1-t)B\| \leq t\|A\| + (1-t)\|B\|.$$

For more on matrix norms, see [14].

**8.2. Perturbation of singular values.** The following perturbative result from matrix analysis is used several times in this manuscript. Let  $A$  and  $B$  be two  $m \times n$  matrices. Let  $k = \min\{m, n\}$ . Let  $\sigma_1, \dots, \sigma_k$  be the singular values of  $A$  in decreasing order and repeated by multiplicities, and let  $\tau_1, \dots, \tau_k$  be the singular values of  $B$  in decreasing order and repeated by multiplicities. Let  $\delta_1, \dots, \delta_k$  be the singular values of  $A - B$ , in any order but still repeated by multiplicities.

**Theorem 8.1.** *For any  $1 \leq p < \infty$ ,*

$$\sum_{i=1}^k |\sigma_i - \tau_i|^p \leq \sum_{i=1}^k |\delta_i|^p,$$

and

$$\max_{1 \leq i \leq k} |\sigma_i - \tau_i| \leq \max_{1 \leq i \leq k} |\delta_i|.$$

The above result follows, for example, from a combination of Theorem III.4.4 and Exercise II.1.15 in [14]. It may also be derived as a consequence of Wielandt's minimax principle [14, Section III.3] or Lidskii's theorem [14, Exercise III.4.3]. The case  $p = 2$  is sometimes called the Hoffman-Wielandt theorem [8, Lemma 2.1.19 and Remark 2.1.20], and the inequality involving the maximum is sometimes called Weyl's perturbation theorem [14, Corollary III.2.6].

**8.3. Bernstein's inequality.** The following inequality is known as 'Bernstein's inequality'.

**Theorem 8.2.** *Suppose that  $X_1, \dots, X_n$  are independent random variables with zero mean, and  $M$  is a constant such that  $|X_i| \leq M$  with probability one for each  $i$ . Let  $S := \sum_{i=1}^n X_i$  and  $v := \text{Var}(S)$ . Then for any  $t \geq 0$ ,*

$$\mathbb{P}(|S| \geq t) \leq 2 \exp\left(-\frac{3t^2}{6v + 2Mt}\right).$$

This inequality was proved by Bernstein [13]. For a discussion of Bernstein's inequality and improvements, see Bennett [12].

**8.4. Talagrand's concentration inequality.** Recall that a median  $m$  of a random variable  $Y$  is a real number such that  $\mathbb{P}(Y \leq m) \geq 1/2$  and  $\mathbb{P}(Y \geq m) \geq 1/2$ . The median may not be unique.

The following concentration inequality is one of the several striking inequalities that are collectively known as 'Talagrand's concentration inequalities'.

**Theorem 8.3.** *Suppose that  $f : [-1, 1]^n \rightarrow \mathbb{R}$  is a convex Lipschitz function with Lipschitz constant  $L$ . Let  $X_1, \dots, X_n$  be independent random variables taking value in  $[-1, 1]$ . Let  $Y := f(X_1, \dots, X_n)$  and let  $m$  be a median of  $Y$ . Then for any  $t \geq 0$ ,*

$$\mathbb{P}(|Y - m| \geq t) \leq 4e^{-t^2/16L^2}.$$

For a proof of Theorem 8.3, see [99, Theorem 6.6].

It is easy to modify Theorem 8.3 to have concentration around the mean instead of the median. Just observe that by Theorem 8.3,  $\mathbb{E}(Y - m)^2 \leq 64L^2$ . Since  $\mathbb{E}(Y - m)^2 \geq \text{Var}(Y)$ , this shows that  $\text{Var}(Y) \leq 64L^2$ . Thus by Chebychev's inequality,

$$\mathbb{P}(|Y - \mathbb{E}(Y)| \geq 16L) \leq \frac{1}{4}.$$

By the definition of a median, this shows that  $\mathbb{E}(Y) - 16L \leq m \leq \mathbb{E}(Y) + 16L$ . Together with Theorem 8.3, this implies that for any  $t \geq 0$ ,

$$(24) \quad \mathbb{P}(|Y - \mathbb{E}(Y)| \geq 16L + t) \leq 4e^{-t^2/2L^2}.$$

The above inequality has a number of uses in the proof of Theorem 2.1.

**8.5. Spectral norms of random matrices.** The following bound on spectral norms of random matrices is a crucial ingredient for this paper. The proof follows from a combinatorial argument of Vu [100] (which is itself a refinement of a classical argument of Füredi and Komlós [47]), together with Talagrand's inequality (24).

**Theorem 8.4.** *Take any two numbers  $m$  and  $n$  such that  $1 \leq m \leq n$ . Suppose that  $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$  is a matrix whose entries are independent random variables that satisfy, for some  $\sigma^2 \in [0, 1]$ ,*

$$\mathbb{E}(a_{ij}) = 0, \quad \mathbb{E}(a_{ij}^2) \leq \sigma^2, \quad \text{and} \quad |a_{ij}| \leq 1 \text{ a.s.}$$

*Suppose that  $\sigma^2 \geq n^{-1+\epsilon}$  for some  $\epsilon > 0$ . Then*

$$\mathbb{P}(\|A\| \geq 2.01\sigma\sqrt{n}) \leq C_1(\epsilon)e^{-C_2\sigma^2n},$$

*where  $C_1(\epsilon)$  is a constant that depends only on  $\epsilon$  and  $C_2$  is a positive universal constant. The same result is true when  $m = n$  and  $A$  is symmetric or skew-symmetric, with independent entries on and above the diagonal, all other assumptions remaining the same. Lastly, all results remain true if the assumption  $\sigma^2 \geq n^{-1+\epsilon}$  is changed to  $\sigma^2 \geq n^{-1}(\log n)^{6+\epsilon}$ .*

*Proof.* First assume that  $m = n$  and  $A$  is symmetric. Note that for any even number  $k$ ,

$$(25) \quad \mathbb{E}\|A\|^k \leq \mathbb{E}(\text{Tr}(A^k)) = \sum_{1 \leq i_1, \dots, i_k \leq n} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}).$$

Consider  $i_1, i_2, \dots, i_{k-1}, i_k, i_1$  as a closed tour of a subset of the vertices of the complete graph on  $n$  vertices (with self-edges included). From the given assumptions about the  $a_{ij}$ 's, it follows that the term  $\mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1})$  is zero if there is an edge that is traversed exactly once. Suppose that each edge in the tour is traversed at least twice. Let  $p$  be the number of distinct vertices visited by the tour. Then the number of distinct edges traversed by the tour is at least  $p - 1$ . Since  $\sigma^2 \leq 1$ ,  $|a_{ij}| \leq 1$ , and  $\mathbb{E}|a_{ij}|^l \leq \sigma^2$  for any  $l \geq 2$ , this shows that

$$(26) \quad |\mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1})| \leq \sigma^{2p-2}.$$

Thus, if  $W(n, k, p)$  is the number of tours of length  $k$  that visit exactly  $p$  vertices and traverse each of its edges at least twice, then

$$(27) \quad \mathbb{E}\|A\|^k \leq \sum_{p=1}^k \sigma^{2p-2} W(n, k, p).$$

Vu [100, equation (5)] proves that if  $p > k/2$  then  $W(n, k, p) = 0$  and if  $p \leq k/2$  then

$$W(n, k, p) \leq n(n-1) \cdots (n-p+1) \binom{k}{2p-2} p^{2(k-2p+2)} 2^{2p-2}.$$

Using this bound, one can proceed as in [100, Section 2] to arrive at the conclusion that if  $k$  is largest even number  $\leq \sigma^{1/3}n^{1/6}$ , then

$$\mathbb{E}\|A\|^k \leq 2n(2\sigma\sqrt{n})^k.$$

Consequently,

$$\mathbb{E}\|A\| \leq (\mathbb{E}\|A\|^k)^{1/k} \leq (2n)^{1/k} 2\sigma\sqrt{n}.$$

This shows that if  $\sigma^2 \geq n^{-1+\epsilon}$  (or if  $\sigma^2 \geq n^{-1}(\log n)^{6+\epsilon}$ ), then there is a constant  $C(\epsilon)$  depending only on  $\epsilon$  such that if  $n \geq C(\epsilon)$  then

$$(28) \quad \mathbb{E}\|A\| \leq 2.005\sigma\sqrt{n}.$$

Since  $a_{ij}$  are independent and  $|a_{ij}| \leq 1$  almost surely for all  $i, j$ , and the spectral norm is a convex Lipschitz function of matrix entries with Lipschitz constant 1 (by the discussion in Section 8.1), therefore one can apply Talagrand's inequality (Theorem 8.3 and inequality (24)) together with (28) and the assumption that  $\sigma^2 \geq n^{-1+\epsilon}$  to conclude that there is a constant  $C(\epsilon)$  such that if  $n \geq C(\epsilon)$  then

$$(29) \quad \mathbb{P}(\|A\| \geq 2.01\sigma\sqrt{n}) \leq C_1 e^{-C_2\sigma^2 n},$$

where  $C_1$  and  $C_2$  are positive universal constants. Replacing  $C_1$  by a large enough constant  $C_1(\epsilon)$ , the condition  $n \geq C(\epsilon)$  may be dropped. It is clear from the argument that it goes through in the skew-symmetric case as well.

Let us now drop the assumption of symmetry, but retain the assumption that  $m = n$ . Let  $a'_{ij} := a_{ji}$ . Then the inequality (25) must be modified to say that for any even  $k$ ,

$$\begin{aligned} \mathbb{E}\|A\|^k &\leq \mathbb{E}(\text{Tr}((A^T A)^{k/2})) \\ &= \sum_{1 \leq i_1, \dots, i_k \leq n} \mathbb{E}(a'_{i_1 i_2} a_{i_2 i_3} a'_{i_3 i_4} a_{i_4 i_5} \cdots a'_{i_{k-1} i_k} a_{i_k i_1}). \end{aligned}$$

As before, the term inside the sum is zero for any tour that traverses an edge exactly once. (In fact, there are more terms that are zero now; a term may be zero even if a tour traverses all of its edges at least twice.) Similarly, the inequalities (26) and (27) continue to hold and therefore so does the rest of the argument.

Lastly, consider the case  $m < n$ . Augment the matrix  $A$  by adding an extra  $n - m$  rows of zeros to make it an  $n \times n$  matrix that satisfies all the conditions of the theorem. Clearly, the new matrix has the same spectral norm as the old one. This completes the proof.  $\square$

**8.6. The estimation lemma.** Suppose that  $A$  and  $B$  are two  $m \times n$  matrices, where  $m \leq n$ . Let  $a_{ij}$  be the  $(i, j)$ th entry of  $A$  and  $b_{ij}$  be the  $(i, j)$ th entry of  $B$ . It is easy to see from definition that

$$\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n (a_{ij} - b_{ij})^2 = \frac{1}{mn} \|A - B\|_F^2 \leq \frac{1}{n} \|A - B\|^2.$$

Thus if  $\|A - B\|$  is small enough, then the entries of  $A$  are approximately equal to the entries of  $B$ , on average. In other words, the matrix  $A$  is an estimate of the matrix  $B$ .

The goal of this section is to show that if in addition to the smallness of  $\|A - B\|$ , we also know that the nuclear norm  $\|B\|_*$  is not too large, it is possible to get a better estimate of  $B$  based on  $A$ .

**Lemma 8.5.** *Let  $A = \sum_{i=1}^m \sigma_i x_i y_i^T$  be the singular value decomposition of  $A$ . Fix any  $\delta > 0$  and define*

$$\hat{B} := \sum_{i: \sigma_i > (1+\delta)\|A-B\|} \sigma_i x_i y_i^T.$$

Then

$$\|\hat{B} - B\|_F \leq K(\delta) (\|A - B\| \|B\|_*)^{1/2},$$

where  $K(\delta) = (4 + 2\delta)\sqrt{2/\delta} + \sqrt{2 + \delta}$ .

*Proof.* Let  $B = \sum_{i=1}^m \tau_i u_i v_i^T$  be the singular value decomposition of  $B$ . Without loss of generality, assume that  $\sigma_i$ 's and  $\tau_i$ 's are arranged in decreasing order. Let  $S$  be the set of  $i$  such that  $\sigma_i > (1 + \delta)\|A - B\|$ . Define

$$G := \sum_{i \in S} \tau_i u_i v_i^T.$$

Note that by the definition of  $\hat{B}$ , the largest singular value of  $A - \hat{B}$  is bounded above by  $(1 + \delta)\|A - B\|$ . In other words,

$$(30) \quad \|A - \hat{B}\| \leq (1 + \delta)\|A - B\|.$$

On the other hand, by Theorem 8.1,

$$\max_{1 \leq i \leq m} |\sigma_i - \tau_i| \leq \|A - B\|.$$

In particular, for  $i \notin S$ ,

$$(31) \quad \tau_i \leq \sigma_i + \|A - B\| \leq (2 + \delta)\|A - B\|,$$

and for  $i \in S$ ,

$$(32) \quad \tau_i \geq \sigma_i - \|A - B\| \geq \delta\|A - B\|.$$

By (31),

$$(33) \quad \|B - G\| \leq (2 + \delta)\|A - B\|.$$

By (30) and (33), we have

$$(34) \quad \|\hat{B} - G\| \leq \|\hat{B} - A\| + \|A - B\| + \|B - G\| \leq (4 + 2\delta)\|A - B\|.$$

Since  $\hat{B}$  and  $G$  both have rank  $\leq |S|$ , the difference  $\hat{B} - G$  has rank at most  $2|S|$ . Using this and (34), we have

$$(35) \quad \|\hat{B} - G\|_F \leq \sqrt{2|S|} \|\hat{B} - G\| \leq (4 + 2\delta)\sqrt{2|S|} \|A - B\|.$$



Next, observe that by (31),

$$(36) \quad \|B - G\|_F^2 = \sum_{i \notin S} \tau_i^2 \leq (2 + \delta) \|A - B\| \sum_{i \notin S} \tau_i \leq (2 + \delta) \|A - B\| \|B\|_*.$$

Combining (35) and (36), we have

$$(37) \quad \begin{aligned} \|\hat{B} - B\|_F &\leq \|\hat{B} - G\|_F + \|B - G\|_F \\ &\leq (4 + 2\delta) \sqrt{2|S|} \|A - B\| + ((2 + \delta) \|A - B\| \|B\|_*)^{1/2}. \end{aligned}$$

Next, note that by (32),

$$\|B\|_* \geq \sum_{i \in S} \tau_i \geq \delta |S| \|A - B\|,$$

and thus

$$(38) \quad |S| \leq \frac{\|B\|_*}{\delta \|A - B\|}.$$

Combining (37) and (38), the proof is complete.  $\square$

**8.7. Finishing the proof of Theorem 2.1.** We will prove the theorem only for the asymmetric model. The only difference in the proofs for the symmetric model and the skew-symmetric model is that we need to use the symmetric and skew-symmetric parts of Theorem 8.4 instead of the asymmetric part.

Throughout this proof,  $C(\epsilon)$  will denote any constant that depends only on  $\epsilon$ , and  $C$  and  $c$  will denote universal constants. The values of  $C(\epsilon)$ ,  $C$  and  $c$  may change from line to line or even within a line.

Note that for all  $i$  and  $j$ ,

$$\mathbb{E}(y_{ij}) = pm_{ij},$$

and

$$(39) \quad \text{Var}(y_{ij}) \leq \mathbb{E}(y_{ij}^2) = p \mathbb{E}(x_{ij}^2) \leq p.$$

Let  $\eta_{ij}$  be the indicator that the  $(i, j)$ th entry is observed. Then  $\eta_{ij}$ 's are independent Bernoulli random variables with expected value  $p$ , and

$$\hat{p} = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n \eta_{ij}.$$

Define two events  $E_1$  and  $E_2$  as

$$\begin{aligned} E_1 &:= \{\|Y - pM\| \leq 2.01\sqrt{np}\}, \\ E_2 &:= \{|\hat{p} - p| \leq 0.001p\}. \end{aligned}$$

By Theorem 8.4,

$$(40) \quad \mathbb{P}(E_1) \geq 1 - C(\epsilon)e^{-cnp}.$$

By Bernstein's inequality (Theorem 8.2), for any  $t \geq 0$ ,

$$\mathbb{P}(|\hat{p} - p| \geq t) \leq 2 \exp\left(-\frac{3mnt^2}{6p + 2t}\right).$$

In particular,

$$(41) \quad \mathbb{P}(E_2) \geq 1 - 2e^{-cmnp}.$$

Let  $\delta$  be defined by the relation

$$(1 + \delta)\|Y - pM\| = 2.02\sqrt{n\hat{p}}.$$

If  $E_1$  and  $E_2$  both happen, then

$$1 + \delta \geq \frac{2.02\sqrt{n\hat{p}}}{2.01\sqrt{np}} \geq \frac{2.02\sqrt{0.999np}}{2.01\sqrt{np}} \geq 1.004.$$

Let  $K(\delta)$  be the constant in the statement of Lemma 8.5. It is easy to see that there is a universal constant  $C$  such that if  $\delta > 0.004$ , then  $K(\delta) \leq C\sqrt{1 + \delta}$ . Therefore by Lemma 8.5, if  $E_1$  and  $E_2$  both happen, then

$$\begin{aligned} \|\hat{p}W - pM\|_F^2 &\leq C(1 + \delta)\|Y - pM\|\|pM\|_* \\ &\leq C\sqrt{n\hat{p}}\|pM\|_* \\ (42) \quad &\leq Cn^{1/2}p^{3/2}\|M\|_*. \end{aligned}$$

By the definition of  $\hat{M}$ , it is obvious that  $|\hat{m}_{ij} - m_{ij}| \leq |w_{ij} - m_{ij}|$  for all  $i$  and  $j$ . Together with (42), this shows that under  $E_1 \cap E_2$ ,

$$\begin{aligned} p^2\|\hat{M} - M\|_F^2 &\leq p^2\|W - M\|_F^2 \\ &\leq C\hat{p}^2\|W - M\|_F^2 \\ &\leq C\|\hat{p}W - pM\|_F^2 + c(\hat{p} - p)^2\|M\|_F^2 \\ &\leq Cn^{1/2}p^{3/2}\|M\|_* + C(\hat{p} - p)^2mn. \end{aligned}$$

Note that  $\mathbb{E}(\hat{p} - p)^2 = p/mn$  and that  $\|\hat{M} - M\|_F^2 \leq 4mn$ . Thus by (40) and (41),

$$\begin{aligned} \mathbb{E}\|\hat{M} - M\|_F^2 &\leq Cn^{1/2}p^{-1/2}\|M\|_* + Cp^{-1} + Cmn(1 - \mathbb{P}(E_1 \cap E_2)) \\ &\leq Cn^{1/2}p^{-1/2}\|M\|_* + Cp^{-1} + C(\epsilon)mne^{-cnp}. \end{aligned}$$

Dividing throughout by  $mn$ , we arrive at the inequality

$$(43) \quad \frac{1}{mn}\mathbb{E}\|\hat{M} - M\|_F^2 \leq \frac{C\|M\|_*}{m\sqrt{np}} + \frac{C}{np} + C(\epsilon)e^{-cnp}.$$

The next goal is to show that

$$(44) \quad \frac{1}{mn}\mathbb{E}\|\hat{M} - M\|_F^2 \leq \frac{C\|M\|_*^2}{mn} + C(\epsilon)e^{-cnp}.$$

First, suppose that  $\|M\|_* > 0.001\sqrt{n/p}$ . Then

$$\frac{\|M\|_*}{m\sqrt{np}} + \frac{1}{np} \leq \frac{C\|M\|_*^2}{mn},$$

and so (44) follows from (43). Therefore assume that  $\|M\|_* \leq 0.001\sqrt{n/p}$ . Then in particular,  $\|M\| \leq 0.001\sqrt{n/p}$ . Therefore if  $E_1 \cap E_2$  happens, then

$$\|Y\| \leq \|Y - pM\| + \|pM\| \leq 2.011\sqrt{np} \leq 2.014\sqrt{n\hat{p}}.$$

This implies that there is no singular value of  $Y$  that exceeds  $2.02\sqrt{n\hat{p}}$ , and therefore  $\hat{M} = 0$ . Consequently,

$$\|\hat{M} - M\|_F^2 = \|M\|_F^2 \leq \|M\|_*^2.$$

Thus, if  $\|M\|_* \leq 0.001\sqrt{n/p}$ , then by (40) and (41),

$$\frac{1}{mn} \mathbb{E} \|\hat{M} - M\|_F^2 \leq \frac{\|M\|_*^2}{mn} + C(1 - \mathbb{P}(E_1 \cap E_2)) \leq \frac{\|M\|_*^2}{mn} + C(\epsilon)e^{-cnp}.$$

This completes the proof of Theorem 2.1. As an end-note, observe that if  $\text{Var}(x_{ij}) \leq \sigma^2$  for all  $i, j$ , then the estimate (39) may be improved to

$$\text{Var}(y_{ij}) = p\text{Var}(x_{ij}) + p(1-p)m_{ij}^2 \leq \max_{(a,b) \in R} (pb + p(1-p)a),$$

where  $R$  is the quadrilateral region

$$\{(a, b) : 0 \leq a \leq 1, 0 \leq b \leq \sigma^2, 0 \leq a + b \leq 1\}.$$

The maximum must be attained at one of the four vertices of  $R$ . An easy verification shows that actually, the maximum is always attained at the vertex  $(1 - \sigma^2, \sigma^2)$ , which gives the upper bound  $\text{Var}(y_{ij}) \leq q := p\sigma^2 + p(1-p)(1 - \sigma^2)$ . This allows us to replace the threshold  $2.02\sqrt{n\hat{p}}$  by  $2.02\sqrt{n\hat{q}}$ , where  $\hat{q} = \hat{p}\sigma^2 + \hat{p}(1 - \hat{p})(1 - \sigma^2)$ . As before we need that  $q \geq n^{-1+\epsilon}$ .

## 9. PROOF OF THEOREM 2.2

Throughout this proof,  $C$  will denote any positive universal constant, whose value may change from line to line.

Take any  $\delta \in [0, m\sqrt{n}]$  and let  $\theta := \delta/(m\sqrt{n})$ . We will first work out the proof under the assumption that  $p < 1/2$ . Under this assumption, three situations are considered. First, suppose that

$$(45) \quad \theta/\sqrt{p} \leq 1 \quad \text{and} \quad m\theta\sqrt{p} \geq 1.$$

Let  $k := \lceil m\theta\sqrt{p} \rceil$ . Clearly,  $k \leq m$ . Let  $M$  be an  $m \times n$  random matrix whose first  $k$  rows consist of i.i.d. Uniform $[-1, 1]$  random variables, and copy this block  $\lceil 1/p \rceil$  times. This takes care of  $k\lceil 1/p \rceil$  rows. (This is okay, since  $k/p \leq m\theta/\sqrt{p} \leq m$  by (45).) Declare the remaining rows, if any, to be zero. Then note that  $M$  has rank  $\leq k \leq m\theta\sqrt{p}$ . Therefore by the inequality (3),

$$\|M\|_* \leq (m\theta\sqrt{p})^{1/2} \|M\|_F \leq (m\theta\sqrt{p})^{1/2} (mn\theta/\sqrt{p})^{1/2} = m\sqrt{n}\theta.$$

Let  $X = M$ . Let  $D$  be our data, that is, the observed values of  $X$ . One can imagine  $D$  as a matrix whose  $(i, j)$ th entry is  $x_{ij}$  if  $x_{ij}$  is observed, and a question mark if  $x_{ij}$  is unobserved. For any  $(i, j)$  belonging to the nonzero portion of the matrix  $M$ ,  $M$  contains  $[1/p]$  copies of  $m_{ij}$ . Since the  $X$ -value at the location of each copy is observed with probability  $p$ , independent of the other copies, and  $p < 1/2$ , therefore the chance that none of these copies are observed is bounded below by a positive universal constant. If none of the copies are observed, then the data contains no information about  $m_{ij}$ . Using this, it is not difficult to write down a formal argument that shows

$$\mathbb{E}(\text{Var}(m_{ij} \mid D)) \geq C.$$

On the other hand, since  $\tilde{m}_{ij}$  is a function of  $D$ , the definition of variance implies that

$$\mathbb{E}((\tilde{m}_{ij} - m_{ij})^2 \mid D) \geq \text{Var}(m_{ij} \mid D).$$

Combining the last two displays, we see that

$$\begin{aligned} \mathbb{E}\|\tilde{M} - M\|_F^2 &\geq \sum_{i=1}^{k[1/p]} \sum_{j=1}^n \mathbb{E}(\tilde{m}_{ij} - m_{ij})^2 \\ (46) \qquad \qquad \qquad &\geq Ck[1/p]n \geq \frac{Cmn\theta}{\sqrt{p}}. \end{aligned}$$

The argument that led to the above lower bound is a typical example of the classical Bayesian argument for obtaining minimax lower bounds, and will henceforth be referred to as the ‘standard minimax argument’ to avoid repetition of details.

Since  $m\theta\sqrt{p} \geq 1$ , it follows from (46) that

$$\frac{\theta}{\sqrt{p}} \geq \frac{1}{mp} \geq \frac{1}{np}.$$

Thus, under (45), there exists  $M$  with  $\|M\|_* \leq \delta$  such that

$$\text{MSE}(\tilde{M}) \geq \frac{C\theta}{\sqrt{p}} + \frac{C}{np}.$$

Next, assume that

$$(47) \qquad \qquad \theta/\sqrt{p} \leq 1 \quad \text{and} \quad m\theta\sqrt{p} < 1.$$

Let  $M$  be an  $m \times n$  matrix whose first row consists of i.i.d. random variables uniformly distributed over the interval  $[-m\theta\sqrt{p}, m\theta\sqrt{p}]$ , and this row is copied  $[1/p]$  times, and all other rows are zero. Then  $M$  has rank  $\leq 1$ , and therefore by (3),

$$\|M\|_* \leq \|M\|_F \leq m\theta\sqrt{p}\sqrt{\frac{n}{p}} = m\theta\sqrt{n}.$$

On the other hand, a standard minimax argument as before implies that for any estimator  $\tilde{M}$ ,

$$\mathbb{E}\|\tilde{M} - M\|_F^2 \geq (m\theta\sqrt{p})^2 \frac{n}{p} = nm^2\theta^2.$$

In particular, under (47), there exists  $M$  with  $\|M\|_* \leq \delta$  such that

$$\text{MSE}(\hat{M}) \geq \frac{C\delta^2}{mn}.$$

Lastly, suppose that

$$(48) \quad \theta/\sqrt{p} > 1.$$

Let  $M$  be an  $m \times n$  matrix whose first  $[mp]$  rows consist of i.i.d. random variables uniformly distributed over  $[-1, 1]$ , and this block is copied  $[1/p]$  times. Then the rank of  $M$  is  $\leq [mp]$ , and so by (48) and (3),

$$\|M\|_* \leq \sqrt{mp}\|M\|_F \leq m\sqrt{np} \leq \theta m\sqrt{n}.$$

Again by a standard minimax argument, it is easy to conclude that for any estimator  $\tilde{M}$ , there exists  $M$  with  $\|M\|_* \leq \delta$  such that

$$\text{MSE}(\tilde{M}) \geq C.$$

This completes the proof when  $p < 1/2$ . Next suppose that  $p \geq 1/2$ . The only place where the assumption  $p < 1/2$  was used previously was for proving that  $\mathbb{E}(\text{Var}(m_{ij}|D)) \geq C$ . This can be easily taken care of by inserting some randomness into the data matrix  $X$ , as follows. First replace  $M$  by  $\frac{1}{2}M$  in all three cases above. This retains the condition  $\|M\|_* \leq \delta$ . Given  $\tilde{M}$ , let  $X$  be the data matrix whose  $(i, j)$ th entry  $x_{ij}$  is uniformly distributed over the interval  $[m_{ij} - 1/2, m_{ij} + 1/2]$ , whenever  $(i, j)$  is the ‘main block’ of  $M$ ; and this value of  $x_{ij}$  is copied  $[1/p]$  times in the appropriate places.

Since the entries of  $M$  are now guaranteed to be in  $[-1/2, 1/2]$ , this ensures that the entries of  $X$  are in  $[-1, 1]$ . Now note that even if  $x_{ij}$  or one of its copies is observed, it gives only limited information about  $m_{ij}$ . In particular, it is easy to prove that  $\mathbb{E}(\text{Var}(m_{ij}|D)) \geq C$  and complete the proof as before.

This complete the proof of Theorem 2.2 for the asymmetric model. For the symmetric model, simply observe that the singular values of any square matrix  $M$  are the same as those of the symmetric matrix

$$\begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}$$

with multiplicity doubled. It is now clear how the minimax arguments for the asymmetric model may be carried over to the symmetric case by considering the same Bayesian models for  $M$  and working with the corresponding symmetrized matrices. For the skew-symmetric case, replace the  $M^T$  by  $-M^T$  in the above matrix.

## 10. PROOF OF THEOREM 2.3

Theorem 2.3 follows from a simple application of the estimation lemma (Lemma 8.5). In the estimation lemma, take  $A = p^{-1}Y$ ,  $B = M$ , and  $\hat{B} = W$ , where  $W$  is defined as in the USVT algorithm, with  $\hat{p} = p$  since  $p$  is known. Let  $\delta$  be as in the estimation lemma, i.e.  $\delta$  solves

$$(1 + \delta)\|p^{-1}Y - M\| = 1.01\gamma.$$

Then, if  $\|p^{-1}Y - M\| < \gamma$ ,

$$\delta = \frac{1.01\gamma}{\|p^{-1}Y - M\|} - 1 > 0.01.$$

There is a universal constant  $C$  such that  $\delta > 0.01$  implies  $K(\delta) \leq C\sqrt{1 + \delta}$ . (As usual,  $C$  will henceforth denote any universal constant, whose value may change from line to line.) Therefore if  $\|p^{-1}Y - M\| < \gamma$ , then

$$\begin{aligned} \|\hat{M} - M\|_F^2 &\leq \|W - M\|_F^2 \\ &\leq K(\delta)^2 \|p^{-1}Y - M\| \|M\|_* \\ &\leq C(1 + \delta) \|p^{-1}Y - M\| \|M\|_* \\ &\leq C\gamma \|M\|_*. \end{aligned}$$

Consequently,  $\|p^{-1}Y - M\| < \gamma$  implies  $\|\hat{M} - M\|_F^2 \leq C\gamma \|M\|_*$ . On the other hand, if  $\|p^{-1}Y - M\| \geq \gamma$ , we have the trivial bound  $\|\hat{M} - M\|_F^2 \leq 4mn$ . Combining, we have

$$\text{MSE}(\hat{M}) \leq \frac{C\gamma \|M\|_*}{mn} + C\epsilon.$$

This completes the proof of Theorem 2.3.

## 11. DISCUSSION AND SUMMARY

This paper introduces a new technique for estimating the entries of a large matrix when the data is a randomly chosen subset of the original entries, possibly with corrupted by noise. There is a large body of literature devoted to this general problem that is summarized in Section 3. The method of this paper works by first scaling the data matrix so that all entries are in  $[-1, 1]$ , then taking the singular value decomposition of the matrix (with zeros substituted for missing values) and then truncating the SVD at the universal threshold of  $2.02\sqrt{n\hat{p}}$ , where  $\hat{p}$  is the proportion of observed values and  $n$  is the maximum of the number of rows and the number of columns. In the final step, the matrix is scaled up by a factor of  $1/\hat{p}$  and the entries that have absolute value greater than 1 are substituted by 1 or  $-1$ , depending on whether they are greater than 1 or less than  $-1$ .

Due to the universality of the threshold, the author calls it the Universal Singular Value Thresholding (USVT) algorithm. The USVT method gives a good estimate when the unknown matrix has a ‘little bit of structure’, where structure is measured by the nuclear norm. Incidentally, the idea of

estimating matrices with low nuclear norm has also appeared very recently in a paper of Davenport et. al. [38].

As is clear from the above description, the USVT algorithm is very simple and fast to implement, with the potential for application to extremely large data matrices. In Theorem 2.2, the USVT estimator is shown to be minimax optimal up to a constant factor in the risk. The algorithm works for matrices that are not necessarily of low rank. On the flip side, the algorithm only gives approximate recovery as opposed to the exact recovery of low rank matrices (under the additional assumption of incoherence) given by the convex relaxation methods of Candès and coauthors [26, 25, 28, 24].

Singular value thresholding has been used before [10, 1] and especially in the important recent papers of Keshavan et. al. [65, 66]; the main new contribution of this paper is the universal threshold of  $2.02\sqrt{n\hat{p}}$ , which eliminates the need for any structural information about the unknown matrix such as rank, etc. The available SVD based algorithms require the underlying matrix to be of low rank and that the rank be known a priori.

The other new contribution of the paper is the solution of many concrete matrix estimation problems in Section 4, including problems related to stochastic blockmodels, distance matrix completion, latent space models, positive definite matrix completion, graph limits and generalized Bradley-Terry models. Many of these problems involve matrices that are not of low rank and are not known to be solvable by any existing method. Simulation results from Section 2.5 and Section 7 indicate that the method may work quite well in practice. Further simulations and numerical comparisons with popular methods have been planned for a forthcoming project.

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